Multiwindow Gabor-type Representations and Signal Representation by Partial Information

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ABSTRACT. Harmonic analysis has been the longest lasting and most powerful tool for dealing with signals and systems which involve both periodic and transient phenomena. Further impact on signal processing was facilitated by what we refer to as representations in combined spaces. These representations, motivated by Gabor's concept of time-frequency information, cells have evolved in recent years into a rich repertoire of Gabor-type windowed Fourier transforms, relating Fourier analysis to the Heisenberg group. Such localized bases or frames are useful in the representation, processing, compression and transmission of speech, images and other natural signals that, by their very nature, are nonstationary. To incorporate scale that lends itself to multiresolution analysis as is the case with wavelets, the Gabor scheme is generalized to multiwindow Gabor frames. The properties of such sequences of functions are characterized by an approach that combines the concept of frames and the Zak Transform. Results on signal representation and reconstruction from partial information in the frequency domain are related to and derived, using relevant results obtained in the time or positional information domain. Some results concerning the representation of Fourier-transformed discrete time (finite) sequences by partial information are rederived by exploiting the duality of the Fourier-Stieltjes transform and its inverse. Results related to discrete signals are extended to continuous one-dimensional signals. Signal and image representation by phase only information is considered also in the context of localized (Gabor) phase, where restoration of magnitude by iterative techniques is much more efficient than in the case of global (Fourier) phase.

1. Introduction

The subject of harmonic analysis is extremely broad, as can be concluded, for example, from the viewpoint of Katzenelson [16], who considers it as the study of objects (functions, measures, etc.) defined on topological groups. Indeed, groups are relevant to Gabor-type representations and wavelet-type transforms [25]. Consider the quasi-regular representation

$$T(g)h(\overline{x}) = kh(g^{-1}\overline{x}), \tag{1.1}$$

where k is a constant that may depend on g, and the independent variable $\overline{x}$ is an n-dimensional vector.

This representation that satisfies the functional homomorphism

$$T(g_1 g_2) = T(g_1)T(g_2)$$

underlines the decomposition of a function h into a package of waves, i.e., wavelets

$$\{T(g_0)h, \cdots, T(g_N)h, \cdots\}.$$

We say that $T(g)$ is a unitary representation with respect to a measure μ if

$$\langle T(g)f,\, T(g)h\rangle_\mu = \langle f, h\rangle_\mu\,. \tag{1.2}$$

A wavelet-type transform is defined, according to the group theoretic approach, as a cross correlation between a signal $f(\overline{x})$ and the wavelets

$$\{T(g)h(\overline{x})\}$$

that is defined as follows:

$$\begin{aligned} C_f(g) &= c\int_M f(\overline{x})T(g)h^*(\overline{x})\mu(\overline{x}) \\ &= c\langle f,\, T(g)h\rangle_\mu\,, \end{aligned} \tag{1.3}$$

where h^* stands for conjugate, $c > 0$ is a normalization constant, h is a template function (or a mother wavelet), and $\mu(\overline{x}) = \rho(\overline{x})d\overline{x}$ is an appropriate invariance measure (if one exists) under the action of the group G.

Given the equation for the analysis, (1.3), we are looking for a way to recover $f(\overline{x})$ from $C_f(g)$, i.e. the equation of the synthesis and the conditions under which it exists.

Most studies of wavelets have been devoted to the special case of the affine group $G = G(A^{-1}, \overline{b})$ of planar scaling and translation. A^{-1} is in this case a diagonal matrix with entries a_i^{-1} and $\overline{b}$ is the translation vector. The quasi-regular unitary representation over $L^2(\mathbb{R}^n)$ is defined in this case by

$$T(g)f(\overline{x}) = kf(A(\overline{x} - \overline{b}))\,, \tag{1.4}$$

where $k = \sqrt{J}$ and $J = |\det(A)|$. Gabor-type representations originate from the Heisenberg group and the associated Weyl operational calculus. As such this representation lacks the scaling that is characteristic of wavelets. Scaling is, of course, important in the analysis of

natural signals. However, the explicit harmonic, or spectral nature of Gabor-type representations is just as important. Thus, it is desirable to incorporate scaling into the Gabor-type representation. This is accomplished by the generalized Gabor scheme [21], [35]. In this paper we focus on signal and image representation by multi-window Gabor-type schemes, where by proper choice of the set of windows we incorporate scaling [35]. We then proceed to present various aspects of signal and image representation by partial information, mostly in the context of harmonic analysis. These two subjects are of utmost importance in signal processing, and have proven to be useful in a wide range of applications (not discussed in this paper). The two interrelated subjects are also instrumental in gaining insight into the structure of natural signals, and of images as such. In both cases we discuss the analysis as well as the synthesis.

2. Multiwindow Gabor-type frames

2.1. *Generalized Gabor-type schemes and the Zak transform*

Many problems in physics and engineering involve the representation and analysis of nonstationary signals and processes. Such problems call for the development and application of sets of functions that are localized in the sense that they rapidly decay in both time (or position in the case of images) and frequency. Such are the wavelets, Gabor functions and the Wigner distribution.

The two widely-studied sequences of the so-called wavelets and Gabor functions are special cases of the following generalized sequence:

$$s_{r,m,n}(x) = g_r(x - na_r)\phi_{r,m}(x), \tag{2.1}$$

where $\{g_r(x)\}$ is a sequence of window functions, $\{a_r\}$ is a set of real numbers and $\{\phi_{r,m}(x)\}$ is a set of kernel functions. In the classical Gabor case [9], there is a single-window function $g(x)$, with $a_r = a$ and b some positive constant, $\phi_{r,m}(x) = e^{j2\pi mbx}$, and $m, n \in \mathbb{Z}$. In the wavelet-type Gabor case, $\phi_{r,m}(x) = 1$, $g_r(x) = b^{-r/2}g(x/b^r)$, $a_r = ab^r$, $r, n \in \mathbb{Z}$, and $g(x)$ is a "mother wavelet" function.

Zibulski and Zeevi [35] analyzed such sequences of functions by developing a matrix algebra approach based on the concept of frames and the Zak transform (ZT). The basic results are presented here without the proofs. The idea of using frames in order to examine the sequence of Gabor functions (and affine wavelets as well) was first proposed and implemented by Daubechies et al. [4]; the so-called Weyl-Heisenberg frames. The ZT of a signal $f(x)$ is defined as follows [14]:

$$(\mathcal{Z}f)(x,u) \triangleq \alpha^{1/2} \sum_{k\in\mathbb{Z}} f\Big[\alpha(x+k)\Big]e^{-j2\pi uk}, \quad -\infty < x, u < \infty, \tag{2.2}$$

with a fixed parameter $\alpha > 0$. The ZT satisfies the following periodic and quasiperiodic relations:

$$(\mathcal{Z}f)(x,u+1) = (\mathcal{Z}f)(x,u),\ (\mathcal{Z}f)(x+1,u) = e^{j2\pi u}(\mathcal{Z}f)(x,u). \tag{2.3}$$

As a consequence of these two relations, the ZT is completely determined by its values over the unit square $(x,u) \in ([0,1)^2)$. This is the essence of this unitary mapping.

Based on the ZT defined by (2.2), we define the Piecewise Zak Transform (PZT) as a vector-valued function $F(x,u)$ of size p [34]:

$$F(x,u) = \Big[F_0(x,u), F_1(x,u) \ldots, F_{p-1}(x,u)\Big]^T, \tag{2.4}$$

where

$$F_i(x,u) = (\mathcal{Z}f)\left(x, u+\frac{i}{p}\right), \quad 0 \leqslant i \leqslant p-1,\ i \in \mathbb{Z}. \tag{2.5}$$

The vector-valued function $F(x,u)$ belongs to $L^2([0,1)\times[0,1/p);\mathbb{C}^p)$, which is a Hilbert space with the inner-product:

$$\langle F, G\rangle = \int_0^1 dx \int_0^{1/p} du \sum_{i=0}^{p-1} F_i(x,u)\overline{G_i(x,u)}.$$

Since the ZT is a unitary mapping from $L^2(\mathbb{R})$ to $L^2([0,1)^2)$, the PZT is a unitary mapping from $L^2(\mathbb{R})$ to $L^2([0,1)\times[0,1/p);\mathbb{C}^p)$. As a consequence we obtain the following inner-product preserving property:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx &= \int_0^1\int_0^1 (\mathcal{Z}f)(x,u)\overline{(\mathcal{Z}g)(x,u)}dxdu \\ &= \int_0^1 dx \int_0^{1/p} du \sum_{i=0}^{p-1} F_i(x,u)\overline{G_i(x,u)}.\end{aligned}$$

This unitary property of the PZT allows us the transformation from $L^2(\mathbb{R})$ to $L^2([0,1)\times[0,1/p);\mathbb{C}^p)$, where issues regarding Gabor-type representations are often easier to deal with and understand.

2.2. *Multi-window Gabor-type expansions*

Generalizing the Gabor scheme, by using several window functions instead of a single-one, the representation of a given signal $f(x) \in L^2(\mathbb{R})$ is given by:

$$f(x) = \sum_{r=0}^{R-1}\sum_{m,n} c_{r,m,n} g_{r,m,n}(x), \tag{2.6}$$

where

$$g_{r,m,n}(x) = g_r(x - na)e^{j2\pi mbx}, \tag{2.7}$$

and $\{g_r(x)\}$ is a set of R distinct window functions. Such a set can incorporate, for example, Gaussian windows of various widths. In this case, with proper oversampling, one can overcome in a way the uncertainty constraint expressed by the limitations of having either a high temporal (spatial) resolution and low frequency resolution or vice versa. This type of a richer Gabor-type representation can be instrumental in applications such as detection of transient signals [8], or in identification and recognition of various prototypic textures and other features in images [22]. Clearly, if $R = 1$ we obtain the single-window Gabor representation.

The characterization of the sequence $\{g_{r,m,n}\}$ can be divided into three categories according to the sampling density of the combined space (the so-called phase space density) defined by $d \triangleq R(ab)^{-1}$: undersampling - $d < 1$, critical sampling - $d = 1$ and oversampling - $d > 1$.

2.3. *Matrix algebra approach for the analysis of frames' properties*

In order to examine the properties of the sequence $\{g_{r,m,n}\}$, we consider the operator:

$$\mathcal{S}f = \sum_{r=0}^{R-1} \sum_{m,n} \langle f, g_{r,m,n} \rangle g_{r,m,n}\ , \tag{2.8}$$

which is a frame operator if $\{g_{r,m,n}\}$ constitutes a frame. For the single-window Gabor scheme, this operator was examined by straightforward methods [5, 13], where application of the ZT was restricted to the case $ab = 1$. In [35], Zibulski and Zeevi show that, if the product ab is a rational number, it might be advantageous to examine this operator in $L^2([0,1) \times [0,1/p); \mathbb{C}^p)$ by using the PZT. A major result of analysis in the PZT domain is the representation of the frame operator as a finite order matrix-valued function, as formulated in the following theorem.

THEOREM 1 (Zibulski and Zeevi [35]). *Let $ab = p/q, \quad p, q \in \mathrm{N}$, and let $\mathcal{S}_z$ be the frame operator of the sequence, which is the PZT of $\{g_{r,m,n}\}$. The action of $\mathcal{S}_z$ in $L^2([0,1) \times [0,1/p); \mathbb{C}^p)$ is given by the following matrix algebra:*

$$(\mathcal{S}_z F)(x,u) = \mathbf{S}(x,u)F(x,u), \tag{2.9}$$

where $\mathbf{S}(x,u)$ is a $p \times p$ matrix-valued function whose entries are given by:

$$\begin{aligned} S_{i,k}(x,u) \quad &= \quad \frac{1}{p} \sum_{r=0}^{R-1} \sum_{l=0}^{q-1} \mathcal{Z}g_r\left(x - l\frac{p}{q}, u + \frac{i}{p}\right) \overline{\mathcal{Z}g_r\left(x - l\frac{p}{q}, u + \frac{k}{p}\right)}; \\ &\qquad i,k = 0, \cdots, p-1, \end{aligned} \tag{2.10}$$

and the vector-valued function $F(x,u)$ is given by (2.4) and (2.5).

Since the PZT is a unitary transform, (2.9) is an isometrically isomorphic representation of $\mathcal{S}$ (2.8).

Using the PZT and the matrix representation of the operator $\mathcal{S}$, Zibulski and Zeevi [35] examined the properties of the sequence $\{g_{r,m,n}\}$ for a rational ab. Since, in the case of undersampling the sequence $\{g_{r,m,n}\}$ is not complete, the results presented next are relevant in the cases of critical sampling and oversampling.

The next theorem examines the completeness of the sequence $\{g_{r,m,n}\}$ in relation to the structure of the matrix-valued function $\mathbf{S}(x,u)$.

THEOREM 2 (Zibulski and Zeevi [35]). *Given* $g_r \in L^2(\mathbb{R}),\ 0 \leqslant r \leqslant R-1$, *and a matrix-valued function* $\mathbf{S}(x,u),\ (x,u) \in ([0,1)\times[0,1/p))$ *as in (2.10), the sequence* $\{g_{r,m,n}\}$ *associated with* $\{g_r\}$, $ab = p/q,\ p,q \in \mathrm{N}$ *is complete if and only if* $\det(\mathbf{S})(x,u) \neq 0$ *a.e. on* $[0,1)\times[0,1/p)$.

The frame bounds of the sequence $\{g_{r,m,n}\}$are determined by the eigenvalues of the matrix-valued function $\mathbf{S}(x,u)$. Let

$$\lambda_{max}(\mathbf{S}) \stackrel{\triangle}{=} \operatorname{ess\,sup}_{(x,u)\in([0,1)\times[0,1/p))} \max_{1\leqslant i\leqslant p} \lambda_i(\mathbf{S})(x,u) \tag{2.11}$$

$$\lambda_{min}(\mathbf{S}) \stackrel{\triangle}{=} \operatorname{ess\,inf}_{(x,u)\in([0,1)\times[0,1/p))} \min_{1\leqslant i\leqslant p} \lambda_i(\mathbf{S})(x,u), \tag{2.12}$$

where $\lambda_i(\mathbf{S})(x,u)$ are the eigenvalues of the matrix $\mathbf{S}(x,u)$. Then, the upper frame-bound $B = \lambda_{max}(\mathbf{S})$, and the lower frame-bound $A = \lambda_{min}(\mathbf{S})$. This result yields the following theorem.

THEOREM 3 (Zibulski and Zeevi [35]).
The sequence $\{g_{r,m,n}\}$ *associated with* $\{g_r\}$, $g_r \in L^2(\mathbb{R}),\ 0 \leqslant r \leqslant R-1$, *and* $ab = p/q,\ p,q \in \mathrm{N}$ *constitutes a frame if and only if* $0 < \lambda_{min}(\mathbf{S}) \leqslant \lambda_{max}(\mathbf{S}) < \infty$.

An alternative approach to determining whether $\{g_{r,m,n}\}$ constitutes a frame is as follows. The following Lemma formulates a necessary and sufficient condition for the existence of an upper frame bound $B < \infty$.

LEMMA 1. *The sequence* $\{g_{r,m,n}\}$ *associated with* $\{g_r\}$, $g_r \in L^2(\mathbb{R}),\ 0 \leqslant r \leqslant R-1$ *and* $ab = p/q,\ p,q \in \mathrm{N}$, *has an upper frame bound* $B < \infty$ *if and only if* $(\mathcal{Z}g_r)(x,u)$ *are all bounded a.e. on* $(0,1]^2$ *(*$\mathcal{Z}g_r \in L^\infty((0,1]^2)$*).*

Theorem 4 below determines whether the sequence $\{g_{r,m,n}\}$ constitutes a frame, when an upper frame bound exists, and which does not necessitate calculation of the eigenvalues of $\mathbf{S}(x,u)$.

THEOREM 4 (Zibulski and Zeevi [35]). *Given* $g_r \in L^2(\mathbb{R}),\ 0 \leqslant r \leqslant R-1$, *such that there exists an upper frame bound* $B < \infty$ *for the sequence* $\{g_{r,m,n}\}$ *associated with* $\{g_r\}$, *and* $ab = p/q,\ p,q \in \mathrm{N}$. *The sequence* $\{g_{r,m,n}\}$ *constitutes a frame if and only if*

$0 < K \leqslant \det(\mathbf{S})(x,u)$ *a.e. on* $[0,1) \times [0,1/p)$*, where the matrix-valued function* $\mathbf{S}(x,u)$ *is as in (2.10).*

The following theorem concerns tight frames and the matrix representation of the frame operator. Recall that a set of functions $\{\psi_n\}$ in a Hilbert space H constitutes a *tight frame* if

$$\sum_n |\langle f, \psi_n \rangle|^2 = A\|f\|^2$$

THEOREM 5 (Zibulski and Zeevi [35]). *Given* $g_r \in L^2(\mathbb{R}),\ 0 \leqslant r \leqslant R-1$*, and a matrix-valued function* $\mathbf{S}(x,u)$ *as in (2.10), the set of functions* $\{g_{r,m,n}\}$ *associated with* $\{g_r\}$*,* $ab = p/q,\ p,q \in \mathrm{N}$ *constitutes a tight frame if and only if* $\mathbf{S}(x,u) = A\mathbf{I}$ *a.e., where* $\mathbf{I}$ *is the identity matrix, and* $A = \frac{q}{p}\sum_{r=0}^{R-1} \|g_r\|^2$*.*

In order to complete the formalism of representation, one has to obtain the dual frame. This can be done by using operator techniques [5, 7]. For the single-window scheme the dual frame $\{\mathcal{S}^{-1}g_{m,n}\}$ is generated by a single dual frame window function [5, 13]. This is, indeed, also the case for the multi-window scheme. Let $\{\gamma_{r,m,n}\}$ denote the dual frame of $\{g_{r,m,n}\}$Then $\{\gamma_{r,m,n}\}$ is generated by a finite set of R dual frame window functions $\{\gamma_r\}$:

$$\gamma_{r,m,n}(x) = \gamma_r(x - na)e^{j2\pi mbx},\ 0 \leqslant r \leqslant R-1,$$

where $\gamma_r = \mathcal{S}^{-1}g_r$. Using the matrix representation (2.9) of the frame operator, the PZT of γ_r, is:

$$\Gamma_r(x,u) = \mathbf{S}^{-1}(x,u)G_r(x,u), \tag{2.13}$$

that is, $\Gamma_r(x,u), G_r(x,u)$ are vector-valued functions in

$$L^2([0,1) \times [0,1/p); \mathbb{C}^p)$$

and $\mathbf{S}^{-1}(x,u)$ is the inverse of the matrix $\mathbf{S}(x,u)$; for example,

$$\mathbf{S}^{-1}(x,u) = [\det(\mathbf{S})(x,u)]^{-1}\mathrm{adj}(\mathbf{S})(x,u).$$

2.4. *The Balian-Low theorem in the case of multi-windows*

In the case of a single-window Gabor scheme with critical sampling, i.e. $ab = 1$, by choosing an appropriate $g(x)$, the sequence $\{g_{r,m,n}\}$ can be complete and constitute a frame, notwithstanding the problems of stability [5]. The sequence $\{g_{r,m,n}\}$ can, however, be complete and not constitute a frame, in which case the representation is unstable. A classical example of such an unstable scheme is the one with a Gaussian window function.

In the case of a single window and critical sampling, the following theorem of Balian and Low indicates that a wide range of well behaved – rapidly decaying and smooth – functions $g(x)$ are excluded from being proper candidates for generators of frames.

THEOREM 6 (Balian-Low Theorem [2, 5, 6]). . *Given* $g \in L^2(\mathbb{R})$*,* $a > 0$ *and* $ab = 1$*, if the sequence* $\{g_{m,n}\}$ *constitutes a frame, then either* $xg(x) \notin L^2(\mathbb{R})$ *or* $g'(x) \notin L^2(\mathbb{R})$*.*

Note that $g'(x) \in L^2(\mathbb{R}) \Leftrightarrow \omega\hat{g}(\omega) \in L^2(\mathbb{R})$, where $\hat{g}$ is the Fourier transform of g.

One of the solutions for this problem is oversampling. In fact, it was proven that in the case of a Gaussian the $\{g_{m,n}\}$ constitutes a frame for all $ab < 1$ [15, 26].

In the case of critical sampling of the multi-window scheme, an interesting question is whether one can overcome the constraint imposed by the Balian-Low Theorem by utilizing several windows. According to the following theorem, if all the windows in the set $\{g_r\}$ are well behaved functions, this is not possible:

THEOREM 7 (Zibulski and Zeevi [35]). *Given $g_r \in L^2(\mathbb{R})$, $0 \leqslant r \leqslant R-1$, $a > 0$ and $R(ab)^{-1} = 1$, if the sequence $\{g_{r,m,n}\}$, as in (2.7), constitutes a frame, then either $xg_r(x) \notin L^2(\mathbb{R})$ or $g_r'(x) \notin L^2(\mathbb{R})$ for some $0 \leqslant r \leqslant R-1$.*

One of the advantages of using more than one window is the possibility of overcoming the constraint imposed by the Balian-Low Theorem on the choice of window functions, by adding an extra window function of proper nature such that the resultant scheme of critical sampling constitutes a frame. Whether one can find a non-well-behaved window function, complementary to a set of well-behaved window functions, such that the inclusive set will generate a frame for critical sampling, depends on the nature of the set of the well-behaved window functions as indicated by the following proposition.

PROPOSITION 1. *Let a set, $\{g_r\}$, $0 \leqslant r \leqslant R-2$, of $R-1$ window functions be given. Denote by $\mathbf{G}^0(x,u)$ the $R-1 \times R$ matrix-valued function with entries $G^0_{r,k}(x,u) = \overline{\mathcal{Z}g_r(x, u+\frac{k}{R})}$, and $\mathbf{P}(x,u) = \mathbf{G}^0(x,u)\mathbf{G}^{0^*}(x,u)$. There exists a window function $g_{R-1}(x)$ such that the inclusive set $\{g_r\}$, $0 \leqslant r \leqslant R-1$ generates a frame for the critical sampling case, if and only if $0 < K \leqslant \det(\mathbf{P})(x,u)$ a.e. on $[0,1) \times [0, 1/R)$.*

An example of $R-1$ well-behaved window functions satisfying $0 < K \leqslant \det(\mathbf{P})(x,u)$ a.e. on $[0,1) \times [0,1/R)$, can be constructed in the following manner. Take a window function $g(x)$ such that the sequence $\{g(x-n/b)e^{j2\pi mx/a}\}$ constitutes a frame for $ab = R/(R-1)$. Note, that this is an oversampling scheme ($1/(ab) = (R-1)/R < 1$) and that there exist, therefore, well-behaved window functions $g(x)$ such that $\{g(x-n/b)e^{j2\pi mx/a}\}$ constitutes a frame (for example the Gaussian function). Construct the following $R-1$ window functions:

$$g_r(x) = g\left(x - \frac{rR}{b(R-1)}\right).$$

Clearly these are well-behaved window functions. Moreover, we obtain

$$(\mathcal{Z}g_r)(x,u) = (\mathcal{Z}g)(x - r\frac{R}{R-1}, u)$$

and the matrix-valued function $\mathbf{G}^0(x,u)$ equals the matrix-valued function $\mathbf{G}(x,u)$ which corresponds to the sequence $\{g(x-na)e^{j2\pi mbx}\}$ (which we denote by $\{g_{r,m,n}\}$). In this case $\{g_{r,m,n}\}$ corresponds to an undersampling scheme. By the duality principle, since

$\{g(x-n/b)e^{j2\pi mx/a}\}$ constitutes a frame, $\{g_{r,m,n}\}$ constitutes a Riesz basis for a sub-space of $L^2(\mathbb{R})$. It can therefore be shown that $0 < K \leqslant \det(\mathbf{P})(x,u)$ a.e. on $[0,1)\times[0,1/R)$.

2.5. *A wavelet-type Gabor scheme*

Consider a generalization of the sequence $\{g_{r,m,n}\}$, where for each window function $g_r(x)$ there is a different set of parameters a_r, b_r:

$$g_{r,m,n}(x) = g_r(x-na_r)e^{j2\pi mb_r x}, \tag{2.14}$$

and the sampling density of the combined space is:

$$d \triangleq \sum_{r=0}^{R-1}(a_r b_r)^{-1}.$$

In the case of the single window, the characterization of the sequence $\{g_{r,m,n}\}$ can be divided into the three categories of undersampling, critical sampling and oversampling according to $d<1,\ d=1,\ d>1$, respectively.

In order to analyze this kind of a scheme, we consider an equivalent one with $a_r = a,\ b_r = b$ for all r [35] and utilize the tools presented in the previous section for the analysis of the sequence properties.

Utilizing the degrees of freedom of choosing a different set of parameters a_r, b_r for each window function g_r, we construct a wavelet-type scheme. Let α, β be positive, real numbers. Given a window function $g(x)$ let

$$g_r(x) = \alpha^{-r/2}g(\alpha^{-r}x).$$

Also, let $a_r = \beta\alpha^r$, and $a_r b_r = R/d$ for all r, where d is the sampling density of the combined space. We then have

$$g_{r,m,n}(x) = \alpha^{-r/2}g(\alpha^{-r}x - n\beta)e^{j2\pi\frac{mxR}{\beta\alpha^r d}}.$$

In this scheme the width of the window is proportional to the translation step a_r, whereas the product $a_r b_r$ is constant. This scheme incorporates scaling characteristics of wavelets and of the Gabor scheme with the logarithmically-distorted frequency axis [21]. However, in contrast with wavelets, this scheme has a finite number of window functions, i.e., resolution levels, and each of the windows is modulated by the infinite set of functions defined by the kernel. Each subset for fixed m can be considered as a finite (incomplete) wavelet-type set, in that it obeys the properties of scaling and translation of a complex prototypic "mother wavelet". For example, if R/d is an integer, the mother wavelet is defined by $g(x)e^{j2\pi\frac{mxR}{\beta d}}$; if not, this definition holds within a complex phase. Thus, all the functions corresponding to each of these mother wavelets are self-similar.

Scaling has been realized as an important property of sets of representation functions which are used in the analysis of natural signals and images. Hence the importance of this type of generalization of the Gabor scheme.

3. Signal representation by partial information

3.1. *Fourier and related transforms*

Let $f(t)$ be a complex function which belongs to $L_1(-\infty,\infty)$ or $L_2(-\infty,\infty)$. The ordinary Fourier transform denoted by $\hat{f}(\omega)$ and its inverse are defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \tag{3.1}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{j\omega t}d\omega\,. \tag{3.2}$$

If $\hat{f}(\omega)$ is zero outside the interval $[a,b]$ then

$$f(t) = \frac{1}{2\pi}\int_{a}^{b} \hat{f}(\omega)e^{j\omega t}d\omega \quad |a|,|b|<\infty\,, \tag{3.3}$$

is called a band-limited function, i.e., $f(t)\in \mathbf{B}^{\Omega}$, where $\mathbf{B}^{\Omega}$ is the space of bandlimited functions.

It is well known from the Paley-Wiener theoremm [23] that such a $f(t)$, with t being interpreted as a complex variable, is an entire function of exponential type (EFET). Due to the duality of the Fourier transform and its inverse, if $f(t)$ is time-limited to $[\alpha,\beta]$, then

$$\hat{f}(\omega) = \int_{\alpha}^{\beta} f(t)e^{-j\omega t}dt\,. \tag{3.4}$$

In this case, with ω being viewed as a complex variable, $\hat{f}(\omega)$ is an EFET. This duality is complete if we change the argument sign and multiply (divide) by 2π. If we define a dual Fourier transform pair $(\hat{f}'(\omega), f'(t))$ by

$$\hat{f}'(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f'(t)e^{j\omega t}dt \tag{3.5}$$

$$f'(t) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-j\omega t}d\omega \tag{3.6}$$

we have perfect duality between time-limited signals and band-limited signals. If $f(t) = f'(t)$, then $\hat{f}(\omega) = 2\pi\hat{f}'(-\omega)$. Thus, if $\hat{f}(\omega)$ is given by

$$\hat{f}(\omega) = P(\omega) + jQ(\omega) = R(\omega)\exp(i\vartheta(\omega)), \quad R(\omega)>0\,, \tag{3.7}$$

where $P(\omega)$ is the real and $Q(\omega)$ the imaginary part, $\hat{f}'(\omega)$ is given by

$$2\pi\hat{f}'(\omega) = \hat{f}(-\omega) = P(\omega) - jQ(\omega)\,. \tag{3.8}$$

Alternatively, $\hat{f}'(\omega)$ can be expressed in polar coordinates

$$2\pi\hat{f}'(\omega) = R(\omega)\exp(-j\vartheta(\omega))\,, \tag{3.9}$$

where $R(\omega)$ is the magnitude and $\vartheta(\omega)$ the phase. All subsequent results relating to time-domain (when the signals are band-limited in frequency) will be used with reference to the frequency-domain (when the signals are of finite duration in time or finite extent in space), with proper interpretation of the duality that exists between frequency and time domain representations.

In general, the signals in the time or frequency domain will not belong to $L_1(-\infty,\infty)$ or $L_2(-\infty,\infty)$. We will use delta functions to describe discrete time signals or discrete Fourier spectra. The application of these functions can be mathematically justified by introducing the Fourier-Stieltjes transform [3], [23],

$$\hat{\theta}(\omega)-\hat{\theta}(0) = \int_{-\infty}^{\infty}\frac{e^{-j\omega t}-1}{-j\omega}d\hat{f}(t) \tag{3.10}$$

$$\hat{f}(T)-\hat{f}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{+j\omega t}-1}{+j\omega}d\hat{\theta}(\omega)\,, \tag{3.11}$$

where $\hat{\theta}(\omega)$ is the cumulative spectrum and $\hat{f}(t)$ the cumulative time function. The Fourier-Stieltjes integrals are properly defined even when the time or frequency functions contain discrete components, while $\hat{\theta}(\omega)$ and $\hat{f}(t)$ are functions of bounded variation in $(-\infty,\infty)$. The duality of the Fourier-Stieltjes transform and its inverse is self-evident by their definitions (3.10), (3.11). $\hat{f}(t)$ is bandlimited in the Fourier-Stieltjes sense if $\hat{\theta}(\omega)$ is constant outside an interval $[a,b]$. In a dual manner we define time-limited signals. In cases where $\hat{\theta}(\omega)$ is unbounded in its variation, more general results should be used, such as Zakai's definition of band- (or time-) limited signals [32], or generalized functions [10]. An important result due to Polya's theorem [17] or Paley-Wiener-Schwartz [10] ensures that if a signal is bandlimited in the Fourier-Stieltjes sense in one domain, its counterpart is an EFET.

A discrete complex time signal is defined by

$$x(t)=\sum_n x_n\delta(t-t_n), \tag{3.12}$$

and its Fourier transform by

$$\hat{x}(\omega)=\sum_n x_n e^{-j\omega t_n}\,. \tag{3.13}$$

Using a transformation of the whole complex plane into itself (ω and z being viewed as complex variables),

$$z=e^{j\omega} \tag{3.14}$$

the generalized z-transform is defined by

$$\hat{x}(z)=\sum_n x_n z^{-t_n}\,. \tag{3.15}$$

The ordinary Fourier sequence and z-transforms are derived through uniformly spaced (normalized) sampling $t_n = n$. The same (within the duality) definitions apply to cases where a discrete signal describes the spectral function.

We use properties of a class of EFET called B-functions. These include all functions of finite bandwidth (according to various definitions of bandwidth). The properties of B-functions are well established by classical theory of entire functions, and a complete characterization of their zeros is known. Furthermore, for each B-function there corresponds a unique expansion into the product of its zeros, determined by the Hadamard factorization [23]:

$$f(s) = cs^m e^{jks} \lim_{r\to\infty} \prod_{|s_n|\leqslant r} (1 - s/s_n)\,, \tag{3.16}$$

where $f(s)$ is a B-function, c a complex constant, m an integer, k a real constant, and s_n are the complex roots of $f(s)$. Convergence of (3.16) is conditional on correct order of the zeros [23]. This form of expansion shows that a band-limited function is uniquely determined (within some parameters) by all zeros.

3.2. *Time and position (i.e. 2D) domains – zero crossings*

Logan's work [17] addresses the issue of whether a one-dimensional bandpass signal can be uniquely determined by its real zero crossings. His work characterizes analytically the class of these interesting signals, and proves that a bandpass signal can be uniquely determined by its real zero crossings (within a multiplicative constant) even if it has complex zeros. Logan's existence theorem defines the conditions that must be satisfied in order to have a representation: 1. The bandpass function and its Hilbert transform should have no zeros in common other than real simple zeros, and 2. The bandwidth of the signal must be less than an octave. These common zeros of the bandpass signal and its Hilbert transform (free zeros in Logan's terminology) are those that may be removed or moved around without destroying the bandpass property of the signal.

The formal conditions for a real bandpass signal representation by its zero crossings are stated in Theorem 8, and for real periodic function, as a special case, in Theorem 9.

THEOREM 8 (Real Bandpass Signal Version (Logan [17])).
Let $h(t) = Re(f(t)e^{j\omega_0 t})$ be a real bandpass function, where $f(t)$ is bandlimited and bounded. $h(t)$ is uniquely determined by its real zero-crossings within a multiplicative constant, if $h(t)$ has no zeros in common with its Hilbert transform $\tilde{h}(t) = I_m(f(t)e^{j\omega_0 t})$ (free zeros) other than real simple zeros, and if the band is less than one octave in width.

THEOREM 9 (Real Periodic Functions (Poggio et al. [20])).
Let $h_s(t)$ be a real periodic bandpass function

$$h_s(t) = \sum_{n=-N}^{N} c_n e^{jnt} \quad c_n = c_{-n}^*\,, \tag{3.17}$$

where c_n is a discrete complex sequence, and $*$ denotes the conjugate operation. The function $h_s(t)$ is uniquely determined by its real zeros within a multiplicative constant provided $h_s(t)$ and its Hilbert transform $\tilde{h}_s(t)$, given by

$$\tilde{h}_s(t) = \sum_{n=-N}^{N} \tilde{c}_n e^{jnt} \quad \tilde{c}_n = -j\ sign(n) c_n\ , \tag{3.18}$$

have no zeros in common except real simple zeros, and $c_n = 0$ for $|n| \leqslant N/2$ (or $(N+1)/2$ when N is odd).

It should be noted that these are existence theorems and as such they define the necessary and sufficient conditions. However, they do not provide an algorithm for signal reconstruction from its zero crossings, nor do they address the important issue of stability. Algorithms for one-dimensional signals were put forward by Voelecker and Requicha [31]. These algorithms were modified by Rotem and Zeevi [24], and applied in reconstruction of images from their zero crossings.

THEOREM 10 (Bandpass Signals in Two Dimensions).
(Rotem and Zeevi [24]) *Let $f(x,y)$ be a real function that satisfies the following conditions:*

(i) $\int\int_{-\infty}^{\infty} f^2(x,y)dx\,dy < \infty$.
(ii) $\int_{-\infty}^{\infty} f^2(x,y_0)dx < \infty$ *for a certain y_0 (such y_0 must exist due to (i)).*
(iii) $\int_{\infty}^{\infty} f^2(x_k,y)dy < \infty$ *for $\{x_n\}$ defined below (see (vii)).*
(iv) *The two-dimensional Fourier transform of $f(x,y)$*

$$\hat{f}(u,v) = \iint_{-\infty}^{\infty} f(x,y)e^{-j(ux+\nu y)}dx\,dy \tag{3.19}$$

satisfies the bandwidth condition:

$$\hat{f}(u,\nu) = 0 \begin{cases} u < -b \cup -a < u < a \cup b < u \\ \nu < -d \cup -c < \nu < c \cup d < u \end{cases} \tag{3.20}$$

where

$$0 < a < b < 2a; \qquad 0 < c < d < 2c\ . \tag{3.21}$$

(v) *For y_0 in (ii), the one-dimensional function of x, $f(x,y_0)$, and its Hilbert transform $\tilde{f}(x,y_0)$ do not have zeros in common except real zeros of degree one.*
(vi) *For $\{x_n\}$ in (iii), the one-dimensional function in y, $f(x_n,y)$, and its Hilbert transform $\tilde{f}(x_n,y)$ do not have zeros in common except real zeros of degree one.*
(vii) *The set of points $\{x_n\}$ constitutes a sampling set for $\mathbf{B}(a,b)$ by having density greater than $(b-a)/\pi$. More precisely: if $N(L)$ is the number of sampling points on the*

interval $(0, L)$, then the sufficient condition is

$$\lim_{L\to\infty} \sup \frac{N(L)}{L} > \frac{b-a}{\pi}\,. \tag{3.22}$$

Then, if there exists another function $g(x, y)$ which also satisfies conditions (i)–(vii), and if

$$sign(g(x, y)) = sign(f(x, y)), \tag{3.23}$$

we have

$$g(x, y) \equiv \alpha f(x, y)\,. \tag{3.24}$$

That is, f and g are identical within a multiple constant. The proof is a direct extension of Logan's theorem [17]. Since the one-dimensional functions $f(x_n, y)$ and $f(x, y_0)$ satisfy Logan's conditions, we can construct the functions along vertical lines $x = x_n$, scale them according to the function on the horizontal line y_0, and then, having a sampling set $\{x_n\}$, the value of $f(x, y)$ is determined (up to a constant) at every point in the plane. Obviously, x and y can be interchanged in conditions (ii)–(vii) and in the proof [24].

THEOREM 11 (Bandpass in One Dimension, Lowpass in Other).
(Rotem and Zeevi [24]) *Let $f(x, y)$ be a real function that satisfies the following conditions:*

(i) $\int\int_{-\infty}^{\infty} f^2(x, y)dx\,dy < \infty$.

(ii) $\int\int_{-\infty}^{\infty} f^2(x, y_n)dx < \infty$ *for $\{y_n\}$ defined below.*

(iii) *The two-dimensional Fourier transform of $f(x, y)$ satisfies*

$$\hat{f}(u, \nu) = 0 \begin{cases} u < -b \cup -a < u < a \cup b < u \\ \nu < -c \cup \nu > c \end{cases} \tag{3.25}$$

where

$$0 < a < b < 2a; \quad 0 < c < \infty\,. \tag{3.26}$$

(iv) *For $\{y_n\}$ in (ii), the one-dimensional function in x, $f(x, y_n)$ and its Hilbert transform $\tilde{f}(x, y_n)$ do not have zeros in common except real zeros of degree one.*

(v) *There exists a straight line ℓ of angle θ to the x axis, where*

$$\tan\theta < \frac{2a-b}{3c} \tag{3.27}$$

(by (iii), $b < 2a$, so $\theta > 0$) such that the one-dimensional function $f(s)$ satisfies along ℓ,

$$\int_{-\infty}^{\infty} f(s)ds < \infty\,. \tag{3.28}$$

(vi) *The function $f(s)$ and its Hilbert transform $\tilde{f}(s)$ do not have zeros in common except real zeros of degree one.*

(vii) *The set of points* $\{y_n\}$ *constitutes a sampling set for* $\mathbf{B}(c)$.

Then, if there exists another function $g(x,y)$ *which also satisfies conditions (i)–(vii), and if*

$$sign(g(x,y)) = sign(f(x,y)), \tag{3.29}$$

we have

$$g(x,y) \equiv \alpha f(x,y)\,. \tag{3.30}$$

In this case, the function is determined on horizontal lines y_n and scaled on the diagonal ℓ. Having $\{y_n\}$, a sampling set for $\mathbf{B}(c)$, we can determine the value for every point between the lines.

The conditions can be interchanged so that the function is determined on multiple diagonals and one horizontal line, or another diagonal angle less than θ relative to the x axis.

The importance of formulating the conditions of free zeros in the case of a two-dimensional signal (e.g. an image), is in the observation that these conditions have to be satisfied only on a sparse set of lines in the plane. Furthermore, if the signal is periodic (and every finite image can be considered as one period of a periodic, infinite signal, thus enabling bandpass conditions to exist) – only a finite number of lines, out of the infinite number of lines in the plane, have to satisfy the free-zero conditions. Thus, even if the strict condition of having no free zero is not realized in an image which satisfies one of the bandpass conditions, the image is still likely to be represented by its zero crossings.

3.2.1. *Reconstruction of images from their zero crossings.* The proposed algorithms directly follow the proofs of Theorems 10 and 11. If the image satisfies the conditions of Theorem 10, the one-dimensional functions on several rows (or columns) are reconstructed, using an iterative algorithm described below. Then, another one-dimensional function is reconstructed along a column (or a row), and used for scaling the previous functions. If the image satisfies the conditions of Theorem 11, the scaling is done along a diagonal at the proper angle. The rest of the image is reconstructed by interpolation.

The reconstruction algorithm is as follows: Given the clipped signal S, pass it through an ideal bandpass filter to yield S_0. Designating the bandpass operation as $B_p\{\cdot\}$, obtain the initial signal

$$S_0 = B_p\{\operatorname{sign} S\} \tag{3.31}$$

and then continue iteratively

$$S_{j+1} = S_n - c[B_p\{\operatorname{sign} S_n\} - S_0]\,. \tag{3.32}$$

The algorithm converges when $\operatorname{sign}(S_i) = \operatorname{sign}(S)$. It can be shown that the process of these successive approximations is contracting. If Logan's conditions for uniqueness are met, then $S_n \equiv \alpha S$, where α is some constant. This algorithm yields good results for most bandpass signals, with the number of iterations varying from a few up to 100. If $\operatorname{sign}(S_i(nT)) = \operatorname{sign}(S(nt))$ for every n, there can still exist another continuous signal $S'(t)$ where $S'(nT) =$

$S_i(nT)$ but $\mathrm{sign}(S'(t)) \not\equiv \mathrm{sign}(S(t))$, so $S'(t) \not\equiv S(t)$ and also $S_i(nT) \not\equiv \alpha S(nT)$. This ambiguity can be viewed as a result of real-zeros jitter that may also change the location of the complex zeros [24]. It should be noted, however, that the dependence of the deviation of the complex zeros on that of the reals zeros is closely related to the question of stability. This issue is dealt with elsewhere [33].

The error analysis indicates some simple ways to reduce the errors in the scaling factors. For example, select a column along which the reconstructed signal converged; locate the maxima of the signal. (On these points the normalized errors are minimal.) Reconstruct the signals along some rows intersecting the selected column at the peak points and scale them according to the peak values. Then reconstruct the signals in the other columns. The intersection points of each column with the rows form two vectors: that of the (as yet unscaled) values of the signal along the column and that of the scaled values of the signals along the rows. Denote the unscaled values along the column by $c(k)$ and the values of the rows by $r(k)$. If there are M rows, then the optimal scaling factor for the column is

$$a = \frac{\sum_{k=1}^{M} c(k)r(k)}{\sum_{k=1}^{M} c^2(k)} \,. \tag{3.33}$$

In this way the effect of reconstruction errors in the columns and the rows tends to cancel out [24].

The same method can be applied to images which are bandpass in one dimension and lowpass in the other (Theorem 11). The scaling is carried out in this case by reconstruction on several diagonals intersecting one column at its maxima, scaling the diagonals, and then scaling the other columns by values of the points of intersection with the diagonals.

In order to have an ideal bandpass (or lowpass) image, the signal should be looked upon as periodic, the actual image being one period (in the two-dimensional sense). The conditions of finite Fourier integrals should in this case be modified for periodic signals.

The bandpass operation can be carried out by calculating the FFT of each column and simply deleting all components out of the passband, and then transposing the image and repeating the operation. This is equivalent to convolving the image with an infinite, periodic bandpass filter.

Prior to performing the sign operation, it is recommended to interpolate the signal so as to reduce the zero-crossings jitter. Interpolation is also needed in creating the diagonals, so that they can be inclined at an angle of less than 45^o. In order to create a bandpass signal along the diagonal, the signal should also be periodic. This is done by continuing the diagonal along some periods of the image until it reaches the starting point. Because of the assumed periodicity of the image, continuing the diagonal means "folding" from bottom to top each time it reaches the last row.

3.3. *Frequency domain – discrete signals*

We now present theorems concerning one-dimensional discrete signal representation by partial information in the frequency domain. These include the cases of phase only, sampled phase, one bit of phase, magnitude, and signed-magnitude. The theorems are quoted as stated in the original papers.

THEOREM 12 (Hayes, Lim and Oppenheim [12]). *Let $x(n)$, $y(n)$ be two finite length real sequences whose z-transforms have no zeros in reciprocal conjugate pairs or on the unit circle. If $\vartheta_x(\omega) = \vartheta_y(\omega)$ for all ω, then $x(n) = \beta y(n)$ for some positive constant β. If $\tan\vartheta(\omega) = \tan\vartheta_y(\omega)$ for all ω, then $x(n) = \beta y(n)$ for some real constant β.*

$\vartheta_x(\omega)$ and $\vartheta_y(\omega)$ are the respective phases of the Fourier sequences of $x(n)$ and $y(n)$.

The results have been extended to complex sequences and the condition prohibiting simple zeros on the unit circle relaxed.

THEOREM 13 (Hayes [11]). *Let $x(n)$, $y(n)$ be complex sequences which are zero outside the interval $[0, N-1]$, with z-transforms having no zeros in conjugate reciprocal pairs or on the unit circle. If $M \geqslant 2N-1$ and $\vartheta_x(\omega) = \vartheta_y(\omega)$ at M distinct frequencies in the interval $[0, 2\pi]$, then $x(n) = \beta y(n)$ for some positive real number β. If $M \geqslant 2N-1$ and $\tan(\vartheta_z(\omega)) = \tan(\vartheta_y(\omega))$ at M distinct frequencies in the interval $[0, 2\pi]$, then $x(n) = \beta y(n)$ for some real number β.*

THEOREM 14 (Hayes, Lim and Oppenheim [12]). *Let $x(n)$ be a real sequence which is zero outside the interval $[0, N-1]$ with $x(0) \neq 0$ and which has a z-transform with no zeros in conjugate reciprocal pairs or on the unit circle. Let $y(n)$ be any real sequence which is also zero outside the interval $[0, N-1]$. If $\vartheta_x(\omega) = \vartheta_y(\omega)$ at $N-1$ distinct frequencies in the interval $(0, \pi)$, then $y(n) = \beta x(n)$ for some positive constant β. If $\tan(\vartheta_x(\omega)) = \tan(\vartheta_y(\omega))$ for $N-1$ distinct frequencies in the interval $(0, \pi)$, then $y(n) = \beta x(n)$ for some real constant β.*

THEOREM 15 (Hayes, Lim and Oppenheim [12]). *Let $x(n)$ and $y(n)$ be two real sequences whose z-transform contain no reciprocal pole-zero pairs and which have all poles, not at $z = \infty$, inside the unit circle. If the magnitude of the Fourier transforms of $x(n)$ and $y(n)$ are equal, then $x(n) = \pm y(n+m)$ for some integer m.*

THEOREM 16 (Thm. 2 of Van Hove, Hayes, Lim, Oppenheim).
[29] *Let $x(n)$ and $y(n)$ be two real, causal (or anticausal), and finite extent sequences with z-transforms which have no zeros on the unit circle. If the signed Fourier magnitudes of the sequences (denoted by $G_x^\alpha(\omega)$ and $G_y^\alpha(\omega)$, respectively, and defined by (3.34) and (3.35)) are equal for all ω, $G_x^\alpha(\omega) = G_y^\alpha(\omega)$ $0 < \alpha < \pi$, then $x(n) = y(n)$. When $\alpha = \pi$ (or 0), and if $G_x^*(\omega) = G_y^*(\omega)$ for all ω and $x(0) = y(0)$, then $x(n) = y(n)$.*

The signed Fourier magnitude $G_x^\alpha(\omega)$ and Fourier magnitude $R_x(\omega)$ are related through

$$G_x^\alpha(w) = S_x^\alpha(\omega)|R_x(\omega)|, \tag{3.34}$$

where the bipolar function $S_x^\alpha(\omega)$, defined by

$$S_x^\alpha(\omega) = \left\{ \begin{array}{ll} +1 & \alpha - \pi \leqslant \vartheta_x(\omega) < \alpha \\ -1 & \text{otherwise} \end{array} \right\} \tag{3.35}$$

incorporates one bit of phase information at each frequency [19], [29].

THEOREM 17 (Oppenheim, Lim, and Curtis [19]). *Let $x(n), y(n)$ be two real, finite extent, causal (or anticausal) sequences with all zeros outside (inside) the unit circle. If*

$$S_x^\alpha(\omega) = S_y^\alpha(\omega), \textit{ for all } \omega,$$

$\alpha = 0, \pi$ then $x(n) = \beta y(n)$, where β is a scale factor. (For $\alpha = \pi/2, \beta$ is a positive factor.)

4. Duality of results in time and frequency domains

Based on the duality of the Fourier-Stieltjes transform and its inverse, relevant well-known results for time domain problems, concerned with unique representation of bandlimited functions by partial information, were applied by Shitz and Zeevi [27] in order to derive results about unique representation of a Fourier transformed discrete or continuous time signal by partial information in the Fourier domain. This approach not only enables the derivation of new results for the continuous signals, but more importantly, also highlights the interrelationship of the problems in the two domains.

Shitz and Zeevi [27] have derived, and in various cases extended the results concerning signal representation by partial information in the frequency domain by application of Logan's results (Theorems 8 and 9), and those of Voelecker [30]. In the sequel we only outline their approach.

Let $x(t)$ be a complex, finite time signal; $x(t)$ equal to zero outside the interval $[0, T)$. Let $\hat{x}(\omega)$ be the Fourier transform of $x(t)$, assumed to exist at least in the Fourier-Stieltjes sense. Then

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_0^T x(t) e^{-j\omega t} dt \tag{4.1}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{x}(\omega) e^{j\omega t} d\omega \tag{4.2}$$

$$\begin{aligned} \hat{x}(\omega) &= P_x(\omega) + jQ_x(\omega) \\ &= R_x(\omega) \exp(i\vartheta_x(\omega)) \quad R_x(\omega) > 0, \end{aligned} \tag{4.3}$$

where $P_x(\omega), Q_x(\omega), R_x(\omega)$, and $\vartheta_x(\omega)$ are, respectively, the real part, imaginary part, magnitude, and phase of $\hat{x}(\omega)$.

4.1. *Signal representation by its Fourier phase*

THEOREM 18 (Shitz and Zeevi [27]). *Let $x(t)$ be a complex finite-time signal the Fourier transform of which $\hat{x}(\nu)$ has no complex conjugate zeros. The signal $x(t)$ is uniquely determined (within a positive multiplicative constant) by its Fourier phase $\vartheta_x(\omega), \forall\omega$. If $\hat{x}(\nu)$ has no conjugate zeros at all (neither complex nor real), then $x(t)$ is uniquely defined (within a real constant) by $\tan(\vartheta_x(\omega))$.*

This version of Theorem 12 is valid for continuous signals. Its discrete version, obtained by using the discrete model

$$x(n) = \sum_n x_n \delta(t - t_n) \quad 0 \leqslant t_n < T \tag{4.4}$$

and by interpreting the free zeros (see Theorem 8) as reciprocal conjugate zeros (in the z-plane), turns out to be an extended version of Theorem 12. In fact, $x(n)$ may under this generalization be a countable sequence provided $0 < t_n < T$ for all n (t_n may, for example, be a convergent sequence) and $\Sigma_n |x_n| < \infty$. This result is evident due to the entireness of $\hat{x}(\nu)$ which contains a countable but not necessarily finite number of zeros (3.16). (Theorem 12 is obtained by choosing $T = N, t_n = N$ and $0 \leqslant n < N$).

Theorem 2 of Hayes, Lim, and Oppenheim [12], which is the no-poles-in-reciprocal-pairs version of Theorem 12, follows the latter and can be derived by using an anticausal signal $x(t) \neq 0$ for $-T < t \leqslant 0$, or alternatively by using the transformation $z = e^{-j\nu}$ instead of $z = e^{j\nu}$.

An alternative and much simpler derivation of Theorem 12 is possible by application of Theorem 9 which already deals with discrete signals (trigonometric sequences). However, under this approach the obtained theorem is restricted to discrete signals.

4.2. *Signal representation by its sampled Fourier phase*

Consider a finite extent complex sequence d_n, which equals zero outside the interval $[0, N-1]$. The discrete Fourier transform $\hat{d}(\omega)$ of d_n is given by

$$\begin{aligned} \hat{d}(\omega) &= \sum_{n=0}^{N-1} d_n e^{-jn\omega} \\ \hat{d}(\omega) &= P_{\hat{d}}(\omega) + jQ_{\hat{d}}(\omega) = R_{\hat{d}}(\omega)\exp(j\vartheta_{\hat{d}}(\omega)) \end{aligned} \tag{4.5}$$

where $P_{\hat{d}}(\omega), Q_{\hat{d}}(\omega), R_{\hat{d}}(\omega)$, and $\vartheta_{\hat{d}}(\omega)$ are the real part, imaginary part, magnitude, and phase of $\hat{d}(\omega)$. The transform is extended to the whole complex plane by replacing ω with $\nu = \omega + ju$.

Similarly to our consideration of the continuous case, we define a new sequence α_n which is time constrained, or "bandpass in time".

$$\alpha_n = 0.5(d_{n-M} + d^*_{-n-M}) \quad M > 0. \tag{4.6}$$

Its Fourier sequence

$$\hat{\alpha}(\nu) = \sum_{n=-(M+N-1)}^{M+N-1} \alpha_n e^{-jn\nu} \tag{4.7}$$

is alternatively written as the sum of two sequences

$$\hat{\alpha}(\nu) = M_{\hat{\alpha}}(\nu) + N_{\hat{\alpha}}(\nu) \tag{4.8}$$

defined by

$$M_a(\nu) = \sum_{n=-M-(N-1)}^{-M} \alpha_n e^{-jn\nu} = 0.5\hat{d}^*(\nu)e^{j\nu M} \tag{4.9}$$

$$N_a(\nu) = \sum_{n=M}^{M+(N-1)} \alpha_n e^{jn\nu} = 0.5\hat{d}(\nu)e^{-j\nu M} \tag{4.10}$$

$$\hat{\alpha}(\nu) = Re(\hat{d}(\nu)e^{-j\nu M}) = R_{\hat{d}}(\nu)\cos(M\nu - \vartheta_{\hat{d}}(\nu))\,. \tag{4.11}$$

$\hat{\alpha}(\omega)$ is real since $d_n = d^*_{-n}$.

The Hilbert transform $\hat{\underset{\sim}{\alpha}}(\nu)$ of $\hat{\alpha}(\nu)$ is defined by

$$\hat{\underset{\sim}{\alpha}}(\nu) = \sum_{n=-M-(N-1)}^{M+(N-1)} \underset{\sim}{\alpha}_n e^{-jn\nu},\ \underset{\sim}{\alpha}_n = i\ sign(n)\alpha_n \tag{4.12}$$

$$\hat{\underset{\sim}{\alpha}}(\nu) = R_{\hat{d}}(\nu)\sin(M\nu - \vartheta_{\hat{d}}(\nu))\,. \tag{4.13}$$

Now we can apply Theorem 9 [20]. The one octave bandwidth requirement is satisfied by choosing $M > (N-1)$. We note that $\hat{\alpha}(-\nu)$ is equivalent to $h(t)$ in the notation of [20]. Interpreting this work in the context of our own notation, it was shown that if ν_0 is a free zero of $\hat{\alpha}(-\nu)$, then both this zero and its conjugate ν_0^* are zeros of $N_{\hat{\alpha}}(\nu)$ [20, eq. (6)]:

$$N_{\hat{\alpha}}(\nu_0) = N_{\hat{\alpha}}(\nu_0^*) = 0\,. \tag{4.14}$$

Derivation of Theorem 13 is straightforward, by application of the formalism provided in [20]. Accordingly $\hat{\alpha}1(\nu)$ is the Fourier sequence of another time bandpass sequence $\alpha 1_n$ which is zero for $|n| \notin [M, M+(N-1)]$ and defined similarly to (4.6), where d_n is replaced by $d1_n$. It was shown in [20] that

$$\begin{aligned} \hat{\alpha}(\omega)\hat{\underset{\sim}{\alpha}}(\nu)1(\omega) - \hat{\underset{\sim}{\alpha}}(\nu)1(\omega)\hat{\alpha}1(\omega) &= R_{\hat{d}}(\omega)R_{\hat{d}1}(\omega)\sin(\vartheta_{\hat{d}}(\omega) - \vartheta_{\hat{d}1}(\omega)) \\ &= \sum_{-(N-1)}^{N-1} e_n e^{-jn\omega}. \end{aligned} \tag{4.15}$$

This equation can be interpreted as an appropriate convolution of two complex sequences of length N, one of them being causal and the other anticausal.

It was shown in [20] that the Vandermonde determinant associated with the real roots of (4.15) is not equal to zero if Logan's conditions are satisfied, so if $\vartheta_{\hat{d}}(\nu) = \vartheta_{\hat{d}1}(\nu)$ for at least $2N-1$ distinct frequencies in the interval $(0, 2\pi)$ then e_n in (4.15) all equal zero. Therefore $\sin(\vartheta_{\hat{d}}(\omega) - \vartheta_{\hat{d}1}(\omega)) = 0$ for all ω or, equivalently, $\tan\vartheta_{\hat{d}}(\omega) = \tan\vartheta_{\hat{d}1}(\omega)$ for all ω.

Logan's condition prohibiting complex free zeros, when expressed in the context of z, dictates a no reciprocal conjugate zeros condition, but still allows zeros on the unit circle. By the fact that $\tan\vartheta_{\hat{d}}(\omega) = \tan\vartheta_{\hat{d}1}(\omega)$ and through Theorem 12 we have $d_n = \beta d1_n$ for some real β, and Theorem 13 follows. It is emphasized, though, that in this case zeros on the unit circle are not allowed except for the case when the points ω_i at which $\vartheta_{\hat{d}}(\omega_i)$ is specified are all the real zeros of $\hat{\alpha}(\omega)$, as discussed in the preceding subsection.

Theorem 5 of Hayes, Lim, and Oppenheim [12] is a special case of Theorem 13 considering a real sequence d_n. The phase $\vartheta_{\hat{d}}(0)$ at $\omega = 0$ is always zero. It is defined in this case for at least N distinct frequencies over the interval $[0, \pi)$, provided $M \geqslant N-1$. Theorem 4 of [12], which is equivalent to Theorem 3 of [12], can be derived in the same way, using anticausal sequences or the complex transformation $z = e^{-j\nu}$.

4.3. *Derivation of Theorem 14*

Theorem 14 in its complex version is derived by directly applying Theorem 9. Through (4.15) we reach the following equality: if $\vartheta_{\hat{d}}(\omega) = \vartheta_{\hat{d}1}(\omega)$, at least for $2N-1$ distinct frequencies in the interval $(0, 2\pi)$, then

$$\frac{\hat{\alpha}(\omega)}{\tilde{\hat{\alpha}}(\omega)} = \frac{\hat{\alpha}1(\omega)}{\tilde{\hat{\alpha}}1(\omega)} \tag{4.16}$$

$$\tan\vartheta_{\hat{d}}(\omega) = \tan\vartheta_{\hat{d}1}(\omega) \forall\omega\,. \tag{4.17}$$

Since $\hat{\alpha}(\nu)$ satisfies Logan's conditions, it can be uniquely determined within a real constant. If $\hat{\alpha}1(\nu)$ also does not possess forbidden free zeros (in Logan's sense), the theorem is proved immediately. If, however, it does possess such free zeros, complex, or real, multiple zeros, then we can identify $\hat{\alpha}1(\omega)$ within a real function $\hat{b}(\omega)$ which is positive or negative for all ω, i.e.,

$$\hat{\alpha}1(\omega) = \hat{b}(\omega)\hat{\alpha}(\omega)\,. \tag{4.18}$$

¿From (4.18) we thus realize that $\alpha 1_n$ is a convolution between α_n and some conjugate symmetric sequence b_n, so that $\alpha 1_n$ cannot be time-bandpass in the same band as α_n, $|n| \in [M, M+(N-1)]$. This is possible only when $\hat{b}(\omega) = \beta$, β being a real constant. Again, if the equality refers to $\vartheta_{\hat{d}}(\omega) = \vartheta_{\hat{d}1}(\omega)$ for distinct $2N-1$ frequencies in $(0, 2\pi)$, the signum function of $\hat{\alpha}1(\omega)$ and $\hat{\alpha}(\omega)$ is uniquely defined, so that $\hat{b}(\omega)$ in (4.18) can be only a positive function and the multiplication factor β is therefore positive. It should be emphasized that

free zeros on the real axis of $\hat{\alpha}(\omega)$ are not permitted according to this theorem, except for the case when the instants ω_i, at which $\vartheta_{\hat{d}}(\omega_i)$ is specified, are all real zero crossing instants of $\hat{\alpha}(\omega)$ (because for zero crossing Logan permits simple free zeros to occur). Theorem 14 is merely the real version of the theorem derived here, restricted to real sequences.

Theorems 6 and 7 (in Hayes et al. [12]) are dual of Theorem 14 here for pole-only sequences. They can be derived using the methods implemented in the case of the dual Theorems 12 and 13 for pole-only sequences.

4.4. *Signal representation by Fourier magnitude*

We turn back to a continuous causal case where $x(t)$ is defined by (4.2) and $\hat{x}(\omega)$ is an analytic signal expressed by

$$\begin{aligned} \hat{x}(\omega) &= P_x(\omega) + jQ_x(\omega) = P_x(\omega) - j\tilde{P}_x(\omega) \\ &= R_x(\omega)\exp(j\vartheta_x(\omega)), \quad R_x(\omega) > 0\,. \end{aligned} \tag{4.19}$$

This is an analytic representation of a lower sideband signal in the frequency domain, with $P_x(\omega)$ being real. Equivalently, $(2\pi)^{-1}\hat{x}(-\omega)$ is an analytic representation of an upper sideband [30], [18].

The question in which cases $(2\pi)^{-1}\hat{x}(-\omega)$ [or equivalently, $\hat{x}(\omega)$] is determined uniquely by its magnitude is equivalent to the problem of determining when a single-side-band (SSB) signal can be unambiguously demodulated (within a sign factor) by an amplitude modulation (AM) detector. We apply Voelecker's [30] results directly to $(2\pi)^{-1}\hat{x}(-\omega)$, and extend Theorem 15 to the continuous case. According to Voelecker, $\hat{x}(-\nu)$ [or $\hat{x}(\nu)$] cannot have zeros in either the upper half plane (UHP) or lower half plane (LHP). Substituting $\nu = \omega + ju$, we have that $u > 0$ and $u < 0$ [or vice versa with respect to $\hat{x}(\nu)$]. Accordingly, we have for continuous signals the following theorem.

THEOREM 19 (Shamai and Zeevi [27]). *Let $x(t)$ be a complex finite-time signal whose Fourier transform $\hat{x}(\nu)$ possesses no zeros either in the lower half-plane (LHP) or in the upper half-plane (UHP). Then $x(t)$ is uniquely determined by the Fourier magnitude $R_x(\omega)$ within a constant time shift and a complex phase factor e^{ja}.*

Representing a discrete time signal by (4.4) and applying the complex transformation $z = e^{j\nu}$, we derive the all zero version of Theorem 15 for complex sequences. The zeros at the origin of z^{-m}, m being an integer, are permitted since they form pure delays. Because these delays cannot be identified by the magnitude, we have the ambiguity $x(n) = \pm y(n+m)$. The formal derivation which includes terms of z^{-m} is obtained directly by defining $x(t)$ to be zero outside an interval (T_0, T), where $T > T_0 > 0$. The pole-zero version of Theorem 15 is derived by resorting to the same reasoning. However, $x(t)$ should in this case be defined as a convolution between two appropriate causal (all-zero) sequences one of which has only the zero terms of Theorem 15 and the other zeros which are the reciprocals of the poles defined in that theorem. The new signal is also an analytic signal, and Theorem 15 follows.

Since Theorem 8 of Hayes, Lim, and Oppenheim [12] is just a mirror version of Theorem 15, it can be derived by the same arguments using the lower half-plane instead of the upper one or vice versa.

4.5. *Signal representation by Fourier signed-magnitude*

Applying well-known results, obtained in the time domain for continuous band-limited functions, Shamai and Zeevi [27] derived the following extended version of Theorem 16.

THEOREM 20 (Shamai and Zeevi [27]).
Let $x(t)$ be a complex signal that vanishes outside the interval $[0, T)$ with $x(0)$ being real. The signal $x(t)$ is uniquely determined (within $x(0)$ for $\alpha = 0, \pi$) by the Fourier signed-magnitude for any $0 \leqslant \alpha \leqslant \pi$, provided the Fourier transform $\hat{x}(\nu)$ contains no multiple real zeros.

4.6. *Signal representation by one bit of Fourier phase*

By direct application of Logan's results [17], a continuous version of Theorem 17, generalized to also incorporate complex signals, was derived.

THEOREM 21 (Shamai and Zeevi [27]).
Let $x(t)$ be a complex signal that vanishes outside the interval $[0, T)$, with $x(0)$ being real. The signal $x(t)$ is uniquely determined (within a positive multiplicative constant for $\alpha \neq 0, \pi$, or within a real constant and within $x(0)$ for $\alpha = 0, \pi$) by the one bit of Fourier phase, provided the Fourier transform $\hat{x}(\nu)$ is zero-free in the closed upper half-plane.

Using the analytic signal

$$\begin{aligned} \hat{x}(\nu)e^{-j\alpha} &= P_x^\alpha(\nu) + jQ_x^\alpha(\nu) \\ &= R_x(\nu)\exp(j(\vartheta_x(\nu) - \alpha)), \end{aligned} \tag{4.20}$$

$f(t)$ in Logan's notation becomes equivalent to $(2\pi)^{-1}\hat{x}(-\nu)e^{-j\alpha}e^{-jT\nu/2}$, in our case, because the Fourier transform (not the inverse) of this function is zero outside the interval $(-T/2, T/2)$. According to Logan's theorem in the context of our notations

$$Re\{(2\pi)^{-1}\hat{x}(-\nu)e^{-j\alpha - jT\nu/2} \cdot e^{j\mu\nu}\}$$

has only real simple zeros if $(2\pi)^{-1}\ \hat{x}(-\nu)e^{-j(jT\nu/2+\alpha)}$ is zero-free in the closed lower half-plane $(\nu = \omega + ju, u \leqslant 0)$.

$P_x^\alpha(\nu)$ is an EFET which is real on the axis $\nu = \omega$ (in fact it is a B-function) and it determines $x(t)$ except perhaps for $x(0)$. If the zeros of $\hat{x}(\nu)$ are all in the closed upper half-plane, $P_x^{\alpha+\pi/2}(\nu)$ has only simple real zeros and it is of course determined within a multiplicative constant by $S_x^\alpha(\nu)$. The latter, in turn, determines the real zero crossings of $Q_x^\alpha(\nu)$ (utilized in the preceding subsection). $S_x^{\alpha+\pi/2}(\nu)$ determines eventually the real zero crossings of $P_x^\alpha(\nu)$. In fact, the multiplicative constant is positive unless $\alpha = 0, \pi$ since, as

was mentioned before, $S_x^\alpha(\nu)$ determines not only the zero crossing instances of $P_x^{\alpha+\pi/2}(\nu)$ but also its sign. In the case of $\alpha = \pi/2$, $P_x^\alpha(\nu)$ also determines $x(0)$ which is assumed in our theorem to be true.

Interpretation of these results for the discrete version is straightforward, by defining $x(t)$ according to (4.4). The conditions with respect to the z variable are found by using $z = e^{j\nu}$. Since $\nu = \omega + ju$ and $u \leqslant 0$, all the zeros in z-notations have to be outside or on the unit circle; those on the unit circle must be simple zeros.

The extended version of Theorem 21 applies to complex sequences, and permits simple zeros on the unit circle; it is valid also for the generalized z-transform, since the results are valid for complex [except for $x(0)$], continuous causal, and finite extent signals. The anticausal case can be derived by the same procedure, using anticausal finite extent complex and continuous signals instead of $x(t)$, or alternatively, in the case of discrete signals, by applying the transformation $z = e^{j\nu}$.

4.7. *Signal and image reconstruction from localized phase*

The subject of one-dimensional signal and image representation by phase information was extended to localized phase. Porat and Zeevi [21] represented images by Gabor phase and showed that such representations by partial information preserve edge information of the original image. Their work was further extended to restoration of magnitude by constrained iterative techniques [1]. Images (and other signals) can be reconstructed from localized (Gabor-type) phase more efficiently than from global (Fourier) phase. One straightforward saving is in the number of computational operations (i.e. computational complexity). The other reason is the much faster convergence rate of the iterative algorithm in the case of localized phase [1]. The reason for a faster convergence rate remained obscured until recently, when Urieli, Porat and Cohen [28] proved the following theorem:

THEOREM 22 (Urieli, Porat and Cohen [28]).
Let L and K be two convex sets whose intersection contains the solution of the POCS (projection onto convex sets) algorithm of iterative orthogonal projections between the two convex sets.

Given a discrete signal $x(n) = [x_0, x_1, \cdots, x_{n-1}]^T$, the optimal convergence angle $\alpha = \pi/4$ is obtained if and only if the $x(n)$ is a geometric sequence, i.e.,

$$x(n) = c[1, q, q^2, \cdots q^{n-1}]^T,$$

where c and q are scalars. For all other signals $\alpha < \pi/4$ (i.e. slower convergence).

How does Theorem 22 relate to the convergence rate reported in [1]? As Urieli et al. [28] observed, if a signal x is sequentially bisected into smaller segments, the ratio of monotonic to nonmonotic sequences increases. Since smaller segments are more likely to be monotonic than larger segments, they are likely to more closely correspond to geometric sequences.

This finding provides an additional good reason for further investigation of natural signals' representation in combined spaces, such as the Gabor-type and wavelet representations.

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