UNIQUENESS SETS FOR THE SPHERICAL TRANSFORM

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A set $E \subset \mathbb{R}^n$ is called a *uniqueness set* (or an *injectivity set*) for the spherical transform if, whenever $f \in C(\mathbb{R}^n)$ and

$$\int_{S(x,r)} f \, d\sigma = 0 \tag{1}$$

for all $x \in E$ and all $r > 0$, f must vanish identically. Here $S(x,r)$ is the $(n-1)$-dimensional sphere of radius r about the point x and σ is area measure on $S(x,r)$.

Problem. *Characterize uniqueness sets.*

In the discussion below, we shall take (solely for simplicity's sake) $n = 2$. In this case, $S(w,r)$ is the circle of radius r about the point $w \in \mathbb{C} = \mathbb{R}^2$ and $d\sigma$ is arclength.

Clearly, if E is the zero set of a (nontrivial) harmonic function u, (1) holds with $f = u$, by the mean-value theorem for harmonic functions. Thus no such set can be a uniqueness set. More generally, suppose that u is a solution to Helmholtz's equation

$$\Delta u + \lambda u = 0, \tag{2}$$

i.e., an eigenfunction of the Laplacian. Then by Weber's relation or Pizzetti's formula (cf. [7, pp.342-343]), we have

$$\int_{S(w,r)} u \, ds = 2\pi r J_0(r\sqrt{\lambda}) u(w),$$

which vanishes whenever $u(w) = 0$. This leads to the following

Conjecture. *A set $E \subset \mathbb{R}^n$* fails *to be a uniqueness set if and only if it is contained in the zero set of some nontrivial solution of* (2).

Examples. 1. Let $E = \{re^{i\theta_j} : r \geq 0,\ j = 1, 2\}$. Then E is a uniqueness set if and only if $\theta_1 - \theta_2 = \alpha\pi$, where α is irrational [4].
2. Let $E = \{r_j e^{i\theta} : 0 \leq \theta < 2\pi,\ j = 1, 2\}$. Then E is a uniqueness set if and only if r_1/r_2 is not a quotient of zeros of the Bessel function J_k for any $k = 0, 1, 2, \ldots$ [6].

Remarks. 1. The (global) structure of zero sets of solutions of (2) is much richer than that of zero sets of harmonic functions.
2. The notion of uniqueness set extends in an obvious way to classes of functions other than $C(\mathbb{R}^n)$; cf. [1], [2], [3], [4], [5]. It is then natural to modify our conjecture to say that a set $E \subset \mathbb{R}^n$ fails to be a uniqueness set for the spherical Radon transform on functions belonging to some class $\mathcal{F}(\mathbb{R}^n)$ if and only if it is contained in the zero set of some nontrivial solution of (2) *which belongs to* $\mathcal{F}(\mathbb{R}^n)$. This fails already for $\mathcal{F}(\mathbb{R}^2) = C_0(\mathbb{R}^2)$, the functions of compact support [2]. Perhaps the modified conjecture is true in those cases in which the class $\mathcal{F}(\mathbb{R}^n)$ contains a sufficiently large supply of eigenfunctions of the Laplacian.

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