Some Unsolved Problems in
Harmonic Analysis
as found in Analytic Number Theory

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1. *Suppose that $T(x) = \sum_{n=0}^{N} c_n \cos n^2 x$, that $T(0) = 1$, and that $T(x) \geq 0$ for all x. Among all such trigonometric polynomials, let $\delta(N)$ denote the minimum possible value of c_0. It is known, via indirect reasoning, that $\delta(N) \to 0$ as $N \to \infty$. How fast, as a function of N, does $\delta(N)$ tend to 0? Like N^{-a}? Like $(\log N)^{-a}$? ...*

2. (Fürstenberg) *Let p and q be positive integers, not powers of the same number. Suppose that x is a real number, and let $\mathcal{S}_p$ be the set of fractional parts of the numbers $p^n x$. Similarly, let $\mathcal{S}_q$ be the set of fractional parts of the numbers $q^n x$. Show that if x is irrational then the sum of the Hausdorff dimensions of the closures of these sets is ≥ 1.*

From this it would follow that 2^n has a 7 in its base 10 expansion, for all large n.

3. *Let $\|\theta\|$ denote the distance from θ to the nearest integer. From a classical theorem of Dirichlet we know that for any real θ and any positive integer N there is an n, $1 \leq n \leq N$, such that $\|n\theta\| \leq 1/(N+1)$. We want something similar, but using only squares: For every real θ and every positive integer N there is an n, $1 \leq n \leq N$, such that $\|n^2\theta\| \leq f(N)$, with $f(N)$ as small as possible.*

Heilbronn showed that one may take $f(N) = N^{-1/2+\varepsilon}$, and more recently A. Zaharescu [16] showed that this can be improved to $f(N) = N^{-4/7+\varepsilon}$. Can this be improved to $f(N) = N^{-1+\varepsilon}$?

4. *Show that if $P(x) = \sum_{j=1}^{k} \alpha_j x^j$, $|\alpha_k - a/q| \leq 1/q^2$, $(a,q) = 1$, then*

$$\sum_{n=1}^{N} e\big(P(n)\big) \ll_k N^{1+\varepsilon}\Big(\frac{1}{q} + \frac{q}{N^k}\Big)^{1/k}.$$

Alternatively, construct examples that violate this.

Even small improvements of existing bounds would be interesting. For example when $k = 3$ and $q \approx N^{3/2}$, derive an upper bound that is $o(N^{3/4})$.

5. *Show that for any positive number B there exist complex numbers $z_1, \ldots, z_N$ such that $|z_n| = 1$ for all n, and $\sum_{n=1}^{N} z_n^{\nu} \ll_B N^{1/2}$ uniformly for $1 \leq \nu \leq N^B$.*

This is known when $B = 2$.

6. *Suppose that $|z_n| \geq 1$ and that $b_n \geq 0$ for all n. Prove that*

$$\max_{1 \leq \nu \leq 2N} \left| \sum_{n=1}^{N} b_n z_n^{\nu} \right| \gg \left(\sum_{n=1}^{N} |b_n|^2 \right)^{1/2},$$

or give a counter-example.

This was proved by Gonek [5] under the stronger hypothesis that $|z_n| = 1$ for all n.

7. *Suppose that N points $\boldsymbol{u}_n$ are given in $\mathbb{T}$. Let $\mathcal{B}$ be a box with one corner at $\boldsymbol{0}$ and the diagonally opposite corner at $\boldsymbol{\alpha}$, and let $D(\boldsymbol{\alpha}) = \sum_{n=1}^{N} \chi_{\mathcal{B}}(\boldsymbol{u}_n) - N \operatorname{area}(\mathcal{B})$ denote the discrepancy of the $\boldsymbol{u}_n$ with respect to this box. Finally, let D denote the supremum of $|D(\boldsymbol{\alpha})|$ over all boxes. It is known that points $\boldsymbol{u}_n$ can be chosen so that $|D(\boldsymbol{\alpha})|$ is uniformly bounded by $c_k (\log N)^{k-1}$. Is it always the case that $\|D\|_{\infty}$ is at least this order of magnitude?*

This is known to be the case when $k = 2$, first proved by W. M. Schmidt [14] in 1971, and then by Halász [6] later by a modified form of Roth's method. But it is not known for larger k.

8. *Let N points be given in $\mathbb{T}^k$, and let $D(\boldsymbol{\alpha})$ denote the associated discrepancy function, as defined above. Show that $\|D\|_1 > c_k (\log N)^{(k-1)/2}$.*

This was proved by Halász [6] when $k = 2$, but the case $k > 2$ is open.

9. *Show that if $|a_n| \leq 1$ for all n, then*

$$\int_0^T \left| \sum_{n=1}^{N} a_n n^{-it} \right|^{2p} dt \ll (T + N^p) N^{p+\varepsilon}$$

uniformly for $1 \leq p \leq 2$.

This is easy to prove when $p = 1$ or $p = 2$. The Density Hypothesis concerning zeros of the Riemann zeta function follows from the above.

10. *Suppose that $0 \leq t_1 < t_2 < \ldots < t_R \leq T$ with $t_{r+1} - t_r \geq 1$ for all r, and $|a_n| \leq 1$ for all n. Show that*

$$\sum_{r=1}^{R} \left| \sum_{n=1}^{N} a_n n^{-it_r} \right|^2 \ll (N + R) N^{1+\varepsilon}.$$

11. *Let $\lambda_1, \ldots, \lambda_N$ be distinct real numbers, and put $\delta_n = \min_{\substack{m \\ m \neq n}} |\lambda_m - \lambda_n|$. Determine the best constant c in the inequality*

$$\sum_{\substack{m,n \\ m \neq n}} \frac{x_m \overline{x_n}}{\lambda_m - \lambda_n} \leq c \sum_{n=1}^{N} \frac{|x_n|^2}{\delta_n} \, .$$

Certainly the best constant is no smaller than π. Montgomery and Vaughan [11] proved the above inequality with $c = 3\pi/2$.

12. **(Littlewood)** *Let* $\|\theta\|$ *denote the distance from* θ *to the nearest integer. Is it true that*

$$\liminf_{n \to \infty} n \, \|n\theta\| \, \|n\phi\| \, = \, 0$$

for any real numbers θ *and* ϕ *?*

13. **(Lehmer)** *The Mahler measure of a polynomial* $P(x) = a_n x^n + \cdots + a_0 = a_n(x - \alpha_1) \cdots (x - \alpha_n)$ *is the quantity*

$$M(P) \, = \, |a_n| \prod_{i=1}^{n} \max(1, |\alpha_i|) \, .$$

Show that there is a constant $c > 1$ *such that if* $P \in \mathbb{Z}[x]$ *and* $M(P) < c$ *then* P *is a product of cyclotomic polynomials (i.e.* $M(P) = 1$ *).*

The least known value of $M(P) > 1$ was found by Lehmer [9] long ago, in an exhaustive search. Lehmer's polynomial $P(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ has arisen many times since. It achieves $M(P) = 1.1762808$.

Let $\theta > 1$ satisfy $\theta^3 - \theta - 1 = 0$. Siegel proved that θ is the least PV number. Smyth [15] proved that if P is irreducible and $M(P) < \theta$ then P is a reciprocal polynomial, that is, $P(x) = x^n P(1/x)$. More recent numerical studies have been conducted by Boyd [2].

14. **(Salem)** *An algebraic integer* $\alpha > 1$ *is called a Salem number if all conjugates of* α *lie in the closed unit disk* $|z| \leq 1$ *, with at least one conjugate on the unit circle. Let* $\mathcal{S}$ *denote the set of all PV numbers, and* $\mathcal{T}$ *the set of all Salem numbers. Salem proved that every member of* $\mathcal{S}$ *is a limit of members of* $\mathcal{T}$ *. Does* $\mathcal{T}$ *have any other limit points? Prove that* 1 *is not a limit point of* $\mathcal{T}$ *. Prove that* $\mathcal{S} \cup \mathcal{T}$ *is closed.*

For basic properties of PV and Salem numbers, see Salem [13] or Bertin, Decomps-Guilloux, Grandet-Hugot, Pathiaux-Delefosse and Schreiber [1]. See also numerous papers of Boyd, and papers cited therein.

15. *Show that there is a constant* $c > 0$ *such that if an algebraic integer* α *and all its conjugates lie in the disk* $|z| \leq 1 + c/n$ *then* α *is a root of unity.*

The best result in this direction thus far is due to Dobrowolski [4], who proved that there is a positive constant c such that if

$$M(\alpha) \, < \, 1 + c \left(\frac{\log \log n}{\log n} \right)^3$$

then α is a root of unity. The value of c has been improved, and the proof simplified, by Cantor and Straus [3], Rausch [12] and Louboutin [10].

16. **(D. J. Newman)** *Let $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N$ be independent random variables with $P(\varepsilon_n = 0) = P(\varepsilon_n = 1) = 1/2$, and put $K(z) = \sum_{n=0}^{N} \varepsilon_n z^n$. Show that there is an absolute constant $c > 0$ such that*
$$P\left(\min_{|z|=1} |K(z)| < 1 \right) > c.$$
Does there exist a choice of the $\varepsilon_n \in \{0, 1\}$ such that $\min_{|z|=1} |K(z)| > c$? What if c is replaced by $c\sqrt{N}$?

17. **(Hardy–Littlewood)** *Suppose that $f \in L^2(\mathbb{T})$, and that $\widehat{f}(k) \neq 0$ only when k is a perfect square. Does it follow that $f \in L^p(\mathbb{T})$ for all $p < 4$?*

A. Córdoba has shown that the answer is in the affirmative if one assumes that the numbers $\widehat{f}(k^2)$ are positive and monotonically decreasing.

18. *Consider a simple closed curve in the plane. A point P is called equichordal if every line through P meets the curve at two points that are a constant distance apart. Can a curve have two distinct equichordal points?*

19. **(Erdős)** *Let $\mathcal{A}$ be an infinite set of positive integers, and let $r(n)$ be the number of ways of writing n as a sum of two members of $\mathcal{A}$. If $r(n) > 0$ for all n, does it follow that $\limsup_{n \to \infty} r(n) = +\infty$?*

20. **(Veech)** *Suppose that α is given, $0 < \alpha < 1/2$. Does there exist a function f, not identically 0, such that f has period 1, is even, $f(0) = 0$, $\sum_{a=1}^{q} f(a/q) = 0$ for every positive integer q, and $f \in \mathrm{Lip}\,(\alpha)$?*

21. *For $P \in \mathbb{C}[z]$ let $\|P\|$ denote the ℓ_1 norm of the coefficients of P. Suppose that A and B are polynomials of degrees a and b, respectively. It is known that the ratio $\|A\|\,\|B\|/\|AB\|$ always lies between 1 and 2^{a+b} provided that $AB \neq 0$. What is the 'usual' or 'average' size of this ratio, as one averages over polynomials in a suitable way?*

22. **(H. Maier)** *Let $\Phi_n(z)$ denote the nth cyclotomic polynomial,*
$$\Phi_n(z) = \prod_{\substack{a=1 \\ (a,n)=1}}^{n} (z - e^{2\pi i a/n}) = \sum_{m=0}^{\phi(n)} c(m, n) z^m,$$
Put $C(n) = \max_m |c(m, n)|$. Show that $(\log C(n))/\log n$ has a limiting distribution as $n \to \infty$.

23. **(Schinzel)** *Let $\Phi_n(z)$ denote the nth cyclotomic polynomial, as above. Show that if n is squarefree then Φ_n has $\gg n^{1/2}$ non-zero coefficients.*

24. **(Balazard)** *Does there exist a Dirichlet series $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ that converges in a half-plane $\Re s > \alpha$ with the property that the equation $D(s) = 0$ has exactly one solution in this half-plane?*

If the Riemann Hypothesis is true then $1/\zeta(s)$ is such Dirichlet series, since then one can take $\alpha = 1/2$ and the only zero is at $s = 1$. The object is to give an unconditional construction.

25. (**Schinzel**) *Suppose that* $f(x) = \sum_{i=0}^{n} a_i x^i$ *with* $a_i \geq 0$ *for all* i, *and that* $f(x)^2 = \sum_{i=0}^{2n} b_i x^i$ *with* $b_i \leq 1$ *for all* i. *Let* $\mathcal{S}(n)$ *denote the set of all such polynomials. It is known that* $\max_{f \in \mathcal{S}(n)} f(1)^2 \sim Cn$. *Show that* $C = 4/\pi$.

Schmidt proved that $4/\pi \leq C \leq 2$ and Schinzel showed that $C \leq 7/4$.

26. (**Ruzsa**) *Let* $\mathcal{A}$ *be a set of positive integers with positive asymptotic density. It follows that gaps between members of the set* $\mathcal{A} - \mathcal{A}$ *are bounded. It does not follow that gaps between members of the set* $\mathcal{A} + \mathcal{A}$ *are bounded. Does it follow that gaps between members of the set* $\mathcal{A} + \mathcal{A} + \mathcal{A}$ *are bounded?*

27. (**Melfi–Erdős**) *Let* $\mathcal{A} = \left\{ \sum \epsilon_i 3^i : \epsilon_i = 0, 1 \right\}$, *and let* $\mathcal{B} = \left\{ \sum \epsilon_i 4^i : \epsilon_i = 0, 1 \right\}$. *Does the set* $\mathcal{A} + \mathcal{B}$ *have positive lower density?*

28. (**Brüdern**) *Let* $\mathcal{P}$ *denote the set of prime numbers. Suppose that a set* $\mathcal{A}$ *is chosen so that* $\mathcal{A} + \mathcal{P}$ *has density* 1. *Let* $A(N)$ *denote the number of members of* $\mathcal{A}$ *not exceeding* N. *How slowly can* $A(N)$ *grow? Certainly one can have* $A(N) < (\log N)^C$. *What is the least possible value of* C?

29. (**Skinner**) *Let* f *be a non-constant polynomial with rational coefficients such that* $f(i)$ *is* 0 *or* 1 *for* $i = 0, 1, \ldots, n$. *How small can the degree of* f *be?*

Clearly $[n/2]$ is a lower bound, and n is an upper bound.

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