Probabilities and Baire's theory in harmonic analysis

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ABSTRACT. This is an expository paper, with an emphasis on the history of the subject. It consists of three main parts:

history, terminology, and examples (Sections 1 and 2),
functions and series (Sections 3, 4, and 5),
thin sets (Sections 6, 7, 8, 9, and 10).

I chose the topics and references according to my own interest, but I decided not to develop what I wrote recently elsewhere [28], so that I was very brief at the end, in particular, on topics I like best.

1. Lebesgue, Baire, and trigonometric series

Lebesgue's measure theory and Baire's category theorem are both a hundred years old [1, 2, 43, 48]. From the very beginning, they were linked with trigonometric series. Baire was inspired by series of continuous functions: His book "Leçons sur les fonctions discontinues" begins with the example of functions defined as sums of everywhere convergent trigonometric series, the main object of Riemann's thesis on trigonometric series [3, 53]. The theory of the Lebesgue integral, as we now know it, was first presented in Lebesgue's book "Leçons sur les séries trigonométriques" [44]. Both books resulted from courses given at the Collège de France. Baire and Lebesgue had been elected "Peccot lecturers," which was an opportunity given to young mathematicians to address younger students. In fact, Denjoy attended Baire's lectures and wrote part of Baire's book (1905), and Fatou attended Lebesgue's lectures and contributed to Lebesgue's book. Fatou wrote his thesis, "Séries trigonométriques et séries de Taylor," at the same time (1906) [14].

The relation between harmonic analysis and integration theory is completely natural, since the first instance of harmonic analysis is the computation of Fourier coefficients using the Fourier formulas, and these formulas involve an integral. The integral in question was not the same for Dirichlet, Riemann, Lebesgue, and Denjoy, but it was either applied (in the case of Dirichlet and Lebesgue) or designed (in the case of Riemann and Denjoy) for the specific purpose of giving a sense to the Fourier formulas. In particular, the second Denjoy totalization was invented specifically to compute the Fourier coefficients of functions considered by Riemann, namely, the sums of everywhere convergent trigonometric series. Riemann stated, without giving a proof, that the coefficients must to tend to zero. Cantor proved this, and moreover, he proved that the coefficients are well defined when the function is given. (This is Cantor's uniqueness theorem.) Denjoy, using the Lebesgue integral, the Baire category theorem, and the transfinite induction of Cantor, provided a way to compute these coefficients. This work was sketched in a Comptes-rendus note in 1921 and was later developed in a series of books between 1941 and 1949 [9, 10].

This Denjoy integral, called "the second totalization" (the first was intended to compute the primitive of the most general derivative [8]), is worth mentioning here because it involves both the Lebesgue integral, which is the prototype of a probability measure, and the Baire category theorem. Furthermore, it performs the most elementary step of harmonic analysis perfectly. However, the Denjoy integral plays no role in the rest of the paper. What is needed from now on is Lebesgue measure on $[0, 1]$, or any equivalent notion of a probability measure, and the Baire theorem expressed as follows:

Baire's category theorem. Let X be a complete metric space, and let G_n, $n = 1, 2, \ldots$, be a sequence of dense, open subsets of X. Then the intersection of the G_n, $\cap_{n=1}^{\infty} G_n$, is dense in X. ($\cap_{n=1}^{\infty} G_n$ is a G_δ set.[1])

Note that Baire could not express his theorem in these terms; the notions of metric space and complete metric space appeared later. They appear at the very beginning of Banach's book, "Théorie des opérations linéaires" (1932), and they are followed immediately by Baire's theorem [4]. The Polish mathematicians of that time were experts in using Baire's theorem in a variety of situations. It is used in Banach's book to prove the Banach–Steinhaus theorem, which was a new way to look at the old "principle of condensation of singularities." In turn, the Banach–Steinhaus theorem is used to show that there exists a continuous function whose Fourier series diverges at a point, the du Bois–Reymond phenomenon [11]. I discovered the power of Baire's theorem in harmonic analysis when Katznelson used it to prove that only analytic functions operate on the Wiener algebra, $A(\mathbb{T}) = \mathcal{F}l^1(\mathbb{T})$, the space of continuous functions on $\mathbb{T}$ whose Fourier series converge absolutely. This result is the converse of the Wiener–Lévy theorem, which asserts that analytic functions do operate on the algebra $A(\mathbb{T})$ [34].

[1] Any countable intersection of open sets is called a G_δ set, and any countable union of closed sets is called an F_δ set

I shall not develop the complete role of measure and topology in harmonic analysis, but rather consider how they are involved in several specific questions.

2. Almost sure and quasi sure; examples

In the terminology of Baire, the countable unions of nowhere dense sets are called sets of the first category. In Bourbaki's terminology, they are called meager sets. They are the analogues of null sets in measure theory.

The complements of sets of the first category (meager sets) are called sets of the second category. They are the analogues of sets of full measure, and they can be defined as sets that contain a dense G_δ set. When a property holds on a dense G_δ set it is called *generic* or *quasi sure*. This is the analogue of almost sure in probability theory. We also say that the property holds *quasi everywhere*, the analogue of almost everywhere, or that it is enjoyed by *quasi all* points, which is the analogue of almost all. The abbreviations q.s. and q.e. are parallel to a.s. and a.e.

The analogy goes further. There is a theorem by Kuratowski and Ulam that is analogous to Fubini's theorem, and it is convenient for us to state it in the following form:

The Kuratowski-Ulam theorem [28]. Let X and Y be Baire spaces (for example, complete metric spaces). Suppose, moreover, that Y is separable, that is, there is a countable base of open sets. Let A be a dense G_δ set in $X \times Y$. For each $x \in X$, define $A(x, \cdot) = \{y \in Y \mid (x, y) \in A\}$. Then $A(x, \cdot)$ is a dense G_δ set in Y for quasi all $x \in X$.

It is important to realize that almost sure and quasi sure properties can be very different, that null sets can be sets of the second category, and that sets of full measure can be meager sets. Here are a few examples.

1. Let $X = \{-1, 1\}^{\mathbb{N}}$. X is both a complete metric space and a probability space, when equipped with the usual probability. Let (c_n), $n \in \mathbb{N}$, be a positive sequence, and consider the series $\sum \pm c_n$ indexed by $(\pm) \in X$. Then

$$\sum \pm c_n \text{ converges a.s. } \iff \sum c_n^2 < \infty,$$
$$\sum \pm c_n \text{ converges q.s. } \iff \sum c_n < \infty.$$

The first result is a theorem of Rademacher, Khintchin, and Kolmogorov. The second is an easy exercise: When $\sum c_n = \infty$, the subset Y of X on which some partial sum of each series $\sum \pm c_n$, $(\pm) \in Y$, exceeds a given number N, is open and dense. Therefore the series is divergent on a dense G_δ set of X. In other words, the series $\sum \pm c_n$ diverges q.s. as soon as any one of them diverges.

2. Let $X = \{-1, 1\}^{\mathbb{N}}$ as before, but now consider the Dirichlet series $\sum_{n=1}^{\infty} \pm n^{-s}$, $s = \sigma + it$. These series converge when $\sigma > 1$. We write $f(s)$ either for their sum when

$\sigma > 1$ or for their analytic continuation. Choosing $(\pm) \in X$, the natural boundary of $f(s)$ is the line $\sigma = 1$ q.s. and the line $\sigma = 1/2$ a.s.

It is more interesting to consider $\sum_{n=1}^{\infty} \pm((2n-1)^{-s} - (2n)^{-s})$, with $(\pm) \in X$. Then the natural boundary of the corresponding function, which we denote by $f_1(s)$, is $\sigma = 0$ q.s. and $\sigma = -1/2$ a.s. Furthermore, the order of $f_1(s)$,

$$\mu(\sigma) = \{\inf a \mid f(\sigma + it) = O(|t|^a),\ |t| \to \infty\},$$

is $(1-\sigma)^+ = \max\{0, 1-\sigma\}$ on $(0,\infty)$ q.s. and $(1/2-\sigma)^+$ a.s. If the difference $(2n-1)^{-s} - (2n)^{-s}$ is replaced by the differences of higher orders,

$$\big((4n-3)^{-s} - (4n-2)^{-s})\big) - \big((4n-1)^{-s} - (4n)^{-s}\big),$$

$$\big((8n-7)^{-s} - (8n-6)^{-s})\big) - \cdots + \big((8n-1)^{-s} - (8n)^{-s}\big),$$

and so on, we obtain functions $f_2(s), f_3(s), \ldots$, that are defined on larger and larger domains. The functions $f_1(s)$, $f_2(s)$, $f_3(s)$,... are all of the form $\sum_{n=1}^{\infty} \pm n^{-s}$ for $\sigma > 1$, although the domain of the coefficients $(\pm)$ becomes restricted as the process continues. Furthermore, $\mu(\sigma) = (1-\sigma)^+$ q.s. and $\mu(\sigma) = (1/2-\sigma)^+$ a.s. on $\mathbb{R}$ where these functions are defined. By taking successive blocks of terms from the series representing the functions $f_1(s)$, $f_2(s)$, $f_3(s), \ldots$, it is possible to define an entire function $f_\infty(s)$ that it is represented by a series of the form $\sum_{n=1}^{\infty} \pm n^{-s}$ when $\sigma > 1$, and such that $\mu(\sigma) = (1-\sigma)^+$ q.s. and $\mu(\sigma) = (1/2-\sigma)^+$ a.s. on $(-\infty,\infty)$. Many variations are possible, and the quasi-sure constructions play an important role in the study of convergence and summability properties of products of Dirichlet series [23, 32, 51].

3. Let $X = \mathbb{T}$ and consider the Hardy-Weierstrass function

$$f(t) = \sum_{n=1}^{\infty} 2^{-n} \cos(2\pi 2^n t).$$

Geza Freud made a careful study of this function. First, as already noticed by Hardy, it is nowhere differentiable. However, $\Delta f(t) = f(t+h) - f(t)$ is $O(|h|)$ as $h \to 0$ for some t, called *slow points*. Actually, the set of slow points is both a null set and a meager set, but it has Hausdorff dimension 1. The modulus of continuity of f is $O(h \log \frac{1}{h})$, meaning that $|\Delta f(t)| < C|h| \log \frac{1}{|h|}$ for all t. However, there are *rapid points* t such that

$$\overline{\lim_{h\to 0}} \frac{|\Delta f(t)|}{|h| \log \frac{1}{|h|}} > 0,$$

and indeed, quasi all t are rapid points. On the other hand,

$$0 < \overline{\lim_{h\to 0}} \frac{|\Delta f(t)|}{|h| \big(\log \frac{1}{|h|} \log\log\log \frac{1}{|h|}\big)^{1/2}} < \infty$$

for almost all t.

We see from these examples that the Baire approach emphasizes divergence, singularities, and large values, while the probabilistic approach favors convergence, smoothing, and regularizing effects. We shall encounter these ideas throughout the paper. However, the first effect of both approaches is to convert rather strange phenomena into familiar ones, to tame monsters, or to generate monsters in a familiar way. I shall illustrate this aspect by considering nondifferentiable continuous functions and noncontinuable Taylor series, before considering other questions.

3. Nowhere-differentiable functions

Nowhere-differentiable continuous functions and noncontinuable Taylor series (i.e., analytic functions whose domain of existence is a disc) were both discovered by Weierstrass. I remind you of what Hermite said about nowhere-differentiable functions in a letter to Stieltjes: "Je me détourne avec horreur et effroi de cette plaie lamentable des fonctions continue qui n'ont pas de dérivée."

Nevertheless, such functions attracted the attention of the physicist Jean Perrin when he observed the trajectories of Brownian particles, and they became part of Wiener's program to build a mathematical theory of Brownian motion, where the realizations are a.s. continuous and nowhere differentiable. Wiener quoted Jean Perrin several times, and his program was only achieved in 1933 with the help of Paley and Zygmund. Nowadays the local behavior of Brownian motion—replete with the concepts of a strong form of nowhere differentiability, modulus of continuity, average behavior (law of the iterated logarithm), rapid points, and slow points—is well understood (see [24] or [29] for references).

The same is true for the Baire point of view. Let us consider $C(I)$, the space of real continuous functions on the interval $I = [0, 1]$. Given any sequence $A_n \to 0$ and an integer ν, the set of $f \in C(I)$ with the property that

$$\exists n > \nu \ : \ \forall k (= 0, 1, \dots, n-1), \quad \frac{1}{n}\Big| f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big)\Big| > A_n$$

is a dense, open subset of $C(I)$, which we denote by G_ν. If $f \in \cap_{\nu=1}^{\infty} G_\nu$, then

$$\overline{\lim_{h\to 0}} \frac{|f(t+h) - f(t)|}{\varphi(|h|)} = \infty \quad \text{for all } t \in [0,1]$$

whenever φ is an increasing function such that $\varphi(1/h) = o(A_n)$ as $n \to \infty$. This is a very strong form of nowhere differentiability. In particular, the Hölder exponent of quasi all f is 0 at every point. Quasi surely in $C(I)$, the multifractal analysis is trivial. To have an interesting multifractal analysis as a generic phenomenon, more restricted classes of functions should be considered, as Stéphane Jaffard has done [17].

4. Random Taylor series: continuation, convergence, and divergence

Weierstrass used a lacunary trigonometric series to construct a continuous nowhere-differentiable function. Later, he constructed a noncontinuable Taylor series using the same idea. Poincaré and Hadamard also used lacunary Taylor series to construct noncontinuable functions. Then, in 1896, Borel issued a strange statement, namely, that in general, a Taylor series is not continuable across its circle of convergence.

Borel had in mind a probabilistic interpretation of this statement. He spoke of arbitrary coefficients (coefficients quelconques) and clearly thought of random, independent phases. In fact, he stated and used a first version of the Borel–Cantelli lemma for that purpose. But Borel's theory of countable probabilities was not yet born, and "in general" could not have a precise meaning in 1896.

Steinhaus gave a rigorous interpretation of probabilistic concepts by means of Lebesgue measure on $I = [0, 1]$. Using binary expansions, he first transferred Lebesgue measure to the space $\{0, 1\}^{\mathbb{N}}$, then to the same space written as $\{0, 1\}^{\mathbb{N}^2}$, and finally to $I^{\mathbb{N}}$. In this way he obtained the so-called Steinhaus sequences $(\omega_n) \in I^{\mathbb{N}}$, for which he proved a zero-one law. Then, by considering random series

(S) $$\sum_{n=0}^{\infty} a_n e^{2\pi i \omega_n} z^n, \quad \overline{\lim_{n\to\infty}}\, a_n^{1/n} = 1,$$

it was not difficult to prove that noncontinuation holds almost surely (1923, 1929, [54, 55]).

Paley and Zygmund proved the same result for Rademacher Taylor series

(R) $$\sum_{n=0}^{\infty} \pm a_n z^n$$

in 1932. They also considered random trigonometric series, and this was the beginning of the theory of random series of functions. In 1933, Wiener joined them and proposed to study Gaussian series, such as the Wiener Fourier series of Brownian motion, in parallel with Steinhaus or Rademacher trigonometric series. To make the notation simpler, I shall consider only Gaussian Taylor series

(G) $$\sum_{n=0}^{\infty} \zeta_n a_n z^n,$$

where the $\zeta_n \in \mathbb{C}$ are independent, normalized Gaussian random variables. When $z = e^{2\pi i t}$, (S), (R), and (G) appear as random trigonometric series with only positive frequencies. The history of these series, culminating with the work of Marcus and Pisier, can be found in the second edition of my book "Some Random Series of Functions" (1985) [24]. Roughly speaking, it was known in 1933 that the series (S), (R), and (G) represent a.s. functions in H^1, H^2, and H^p with $p < \infty$, if and only if $\sum |a_n|^2 < \infty$. Marcus and Pisier established that uniform convergence in the closed unit disc has the same probability, 0 or 1, for series

(S), (R), and (G), and in this way reduced the study of uniform convergence of Rademacher Taylor series to the same, but more tractable, problem involving Gaussian processes.

Nevertheless, not all results are the same for the series (S), (R), and (G). An interesting example is the problem of divergence everywhere at the boundary of the unit disc, which was considered by Dvoretzky and Erdős [23]. The condition

$$(*) \qquad \overline{\lim_{n\to\infty}} \frac{1}{\log n} |a_1 + a_2 + \cdots + a_n| > 0$$

implies that each of the series (S), (R), and (G) diverges everywhere on the circle $|z| = 1$ [24]. But there are series (R) that converge somewhere, while the corresponding (G) diverges everywhere. It suffices to choose

$$\text{(R)} : \sum_{j=0}^{\infty} \pm \alpha_j z^{4^j} \quad \text{and} \quad \text{(G)} : \sum_{j=0}^{\infty} g_j \alpha_j z^{4^j},$$

where $g_j = \zeta_{4^j}$, $\alpha_j = o(1)$, $\lim_{j\to\infty} \alpha_j \sqrt{\log j} = \infty$, and $\overline{\lim}_{j\to\infty} \alpha_j |g_j| = \infty$ a.s. This example also shows that $\log n$ cannot be replaced by any $o(\log n)$ in condition (*). (See the Paley–M. Weiss theorem in Section 7.)

Divergence everywhere raises apparently difficult questions. Is it true that the probability of everywhere divergence increases when we increase the moduli of the coefficients? The corresponding questions for convergence everywhere, convergence almost everywhere, and divergence almost everywhere (decreasing or increasing the moduli of the coefficients, according to the case) have positive answers. The question can be considered for other series of functions, for example, for random trigonometric series or for random Walsh series.

Other problems arise when we consider the properties of analytic functions defined by the series (R), (S), or (G). Does their range cover the plane $\mathbb{C}$ and by how much? The answer is fairly precise for (G), but it is still incomplete for (R) and (S) (references can be found in [27], p. 267).

5. Generic trigonometric and power series

Let me turn to another interpretation of Borel's statement on the noncontinuation of Taylor series as a generic phenomenon. Let $H(D)$ denote the space of analytic functions defined on the unit disc $D = \{z \mid |z| < 1\}$ and endowed with the topology of uniform convergence on compact subsets of D. Let X be a complete metric space consisting of analytic functions defined on D such that the mappings $f \mapsto f^{(n)}(z)$ are continuous from X to $\mathbb{C}$ whenever

$n \in \mathbb{N}$ and $z \in D$. Given $a \in D$, $r > 0$, and $A \in \mathbb{N}$, let

$$G(a, r, A) = \Big\{ f \in X \ \Big| \ \sum_{n=0}^{\infty} \frac{1}{n!} |f^{(n)}(a)| r^n > A \Big\},$$

$$J(a, r) = \bigcap_{A>0} G(a, r, A) = \Big\{ f \in X \ \Big| \ \sum_{n=0}^{\infty} \frac{1}{n!} |f^{(n)}(a)| r^n = \infty \Big\}.$$

The $G(a, r, A)$ are open, and $J(a, r)$ is a G_δ set. Define

$$J = \bigcap_{a,r} J(a, r),$$

where the coordinates of a and r are rational and $r > 1 - |a|$. Then J is also a G_δ set, and the functions f belonging to J are not continuable across the circle $|z| = 1$.

It follows that quasi all f in X are noncontinuable as soon as, given any domain Ω strictly larger than D, the set of f that cannot be continued analytically on Ω is dense in X. In particular, functions represented by (R), (S), or (G), as well as functions belonging to $H(D)$, are quasi surely not continuable across the circle. This result is due to Kierst and Szpilrajn for $H(D)$ [39], and it can be extended in a number of ways, by replacing D with any other domain in $\mathbb{C}$ or $\mathbb{C}^d$, in the following form: Either all functions belonging to the space under consideration can be extended in the same larger domain, or quasi all functions are noncontinuable [28].

Article [28] contains many generic properties of trigonometric and Taylor series. Let me quote a few of them.

1. $X = C(\mathbb{T})$. The partial sums $S_n(f, t)$ of the Fourier series of quasi all f satisfy

$$\overline{\lim_{n\to\infty}} \frac{1}{\omega_n} S_n(f, t) = \infty \quad \text{quasi everywhere}$$

whenever $0 < \omega_n = o(\log n)$ as $n \to \infty$.

2. $X = L^1(\mathbb{T})$. The partial sums $S_n(f, t)$ of quasi all f satisfy

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} S_n(f, t) = \infty \quad \text{everywhere}$$

whenever $0 < \lambda_n = o\Big(\frac{\sqrt{\log n}}{\sqrt{\log\log n}}\Big)$ as $n \to \infty$. (This is the generic Konyagin theorem [40].)

3. $X = H(D)$. Given any open subset of D, say Δ, such that the boundaries of D and Δ have at least one point in common, quasi all f satisfy

$$f(e^{i\theta}\Delta) = \mathbb{C} \quad \text{for all } \theta \in \mathbb{R}.$$

4. $X = H(D)$. The Taylor series of quasi all f satisfy the Nestoridis universality property: Given any compact set K in $\{z \mid |z| \geqslant 1\}$ such that $\mathbb{C} \setminus K$ is connected, and given

any function F that is continuous on K and analytic in its interior, there exists a sequence of partial sums that converge to F uniformly on K [47]. As a corollary, one can show that every continuous function on $\{z \mid |z| = 1\}$ is a pointwise limit of some sequence of partial sums.

5. $X = c_0(\mathbb{N})$. Quasi all Fourier Taylor series $\sum_{n=0}^{\infty} a_n e^{2\pi int}$, $(a_n) \in X$, have the Menchoff universality property: Every Lebesgue-measurable function $F(e^{2\pi it})$ is a limit of some sequence of partial sums almost everywhere [31].

6. Thin sets and function spaces

The last part of this paper concerns thin sets in $\mathbb{Z}$ and $\mathbb{T}$. "Thin" refers to properties related to trigonometric series. Furthermore, the thin sets we consider in $\mathbb{Z}$ are lacunary, and the thin sets we consider in $\mathbb{T}$ are closed and have zero Lebesgue measure.

Probability methods have proven to be quite efficient for the study of thin sets in $\mathbb{Z}$, however, so far, I know of no use of the Baire theory in this context. On the other hand, both probability methods and Baire's method are actively used for exhibiting properties of thin sets in $\mathbb{T}$. As one would expect, the results obtained by the two methods are quite different and often go in opposite directions. A way to relate them was discovered by Körner; it will be sketched at the end of the paper.

The first use of thin sets of integers was made by Hadamard for studying lacunary Taylor series. Assuming that (λ_n) is an increasing sequence of positive integers such that (*) $\lambda_{n+1}/\lambda_n > q > 1$ for all n, he showed that all series $\sum_{n=0}^{\infty} a_n z_n^{\lambda_n}$ with a finite, nonzero radius of convergence are noncontinuable. Condition (*) is far from being the best for this kind of result. However, (*) plays a role in a number of questions in harmonic analysis. It is called the Hadamard lacunary condition.

Let me proceed with the definitions. $C(\mathbb{T})$, $L^p(\mathbb{T})$ $(1 \leqslant p \leqslant \infty)$, $c_0(\mathbb{Z})$, and $l^p(\mathbb{Z})$ $(1 \leqslant p \leqslant \infty)$ have the usual meaning. $C_{as}(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ consisting of functions $f \sim \sum_{n\in\mathbb{Z}} \hat{f}_n e^{2\pi int}$ such that $\sum_{n\in\mathbb{Z}} \pm \hat{f}_n e^{2\pi int}$ represents a.s. a continuous function. Then, according to Marcus and Pisier, the same is true for the Gaussian Fourier series $\sum_{n\in\mathbb{Z}} \zeta_n \hat{f}_n e^{2\pi int}$, where the ζ_n are i.i.d. normalized Gaussian variables, and $C_{as}(\mathbb{T})$ is a Banach space using either of the equivalent norms

$$E\Big\| \sum_{n\in\mathbb{Z}} \pm \hat{f}_n e^{2\pi int} \Big\|_{C(\mathbb{T})} \quad \text{or} \quad E\Big\| \sum_{n\in\mathbb{Z}} \zeta_n \hat{f}_n e^{2\pi int} \Big\|_{C(\mathbb{T})}.$$

The Pisier algebra is $C(\mathbb{T}) \cap C_{as}(\mathbb{T})$. $A(\mathbb{T}) = \mathcal{F}l^1(\mathbb{Z})$ is the Wiener algebra, which is the subspace of those functions in $C(\mathbb{T})$ whose Fourier series converge absolutely (see Section 1). The space of Radon measures on $\mathbb{T}$ is denoted by $M(\mathbb{T})$, and the cone of probability measures is denoted by $M_1^+(\mathbb{T})$. The pseudomeasures are linear forms on $A(\mathbb{T})$, in other words, Schwartz distributions on $\mathbb{T}$ whose Fourier coefficients are bounded. The space of pseudomeasures, $\mathcal{F}l^\infty(\mathbb{Z})$, is denoted by $PM(\mathbb{T})$. Pseudofunctions are pseudomeasures whose

Fourier coefficients tend to 0. The space of pseudofunctions, $\mathcal{F}c_0(\mathbb{Z})$, is denoted by $PF(\mathbb{T})$. The Rajchman measures are at the same time measures and pseudofunctions. We write

$$M_0(\mathbb{T}) = M(\mathbb{T}) \cap PF(\mathbb{T}) = \{\mu \in M(\mathbb{T}) \mid \hat{\mu}(n) = o(1),\ n \to \pm\infty\}.$$

We shall also be interested in the classes $M_\alpha(\mathbb{T})$, $0 \leqslant \alpha \leqslant 1$, defined as

$$M_\alpha(\mathbb{T}) = \{\mu \in M(\mathbb{T}) \mid \hat{\mu}(n) = o(|n|^{-\alpha/2}),\ n \to \pm\infty\}.$$

7. Sidon and Zygmund sets

Given any family $F(\mathbb{T})$ of functions or distributions defined on $\mathbb{T}$ and any subset Λ of $\mathbb{Z}$, we write F_Λ for the subfamily of $F(\mathbb{T})$ whose elements have their spectrum in Λ. Thus, $f \in F_\Lambda$ if and only if

$$f \sim \sum_{n\in\Lambda} \hat{f}_n e^{2\pi i n t}.$$

Here is the first and most important definition of a thin set of integers.

D1. Λ is a Sidon set means that $C_\Lambda = A_\Lambda$.

Equivalent definitions, known already in the 1930s, are $\widehat{M}|_\Lambda = \widehat{PM}|_\Lambda$ and $\widehat{L^1}|_\Lambda = \widehat{PF}|_\Lambda$. (Here we write $\widehat{F}|_\Lambda$ for the space of restrictions to Λ of Fourier transforms of elements of $F(\mathbb{T})$; then $\widehat{PM}|_\Lambda = l^\infty(\Lambda)$ and $\widehat{PF}|_\Lambda = c_0(\Lambda)$.) The structure of Sidon sets is not yet clarified. Probability methods have helped, and there have been two approaches: 1) the use of random functions with a given spectrum, and 2) the use of random sets of integers.

The first method was introduced in 1957 [18, 19] and gave the following result: If Λ is a Sidon set, there exists a $K > 0$ such that given positive integers n and s, every "net" of the form

$$N(a_1, a_2, \ldots, a_n;\ s) = \left\{a_1 m_1 + a_2 m_2 + \cdots + a_n m_n \ \middle|\ m_j \in \mathbb{Z}, \sum_{j=1}^{n} |m_j| < 2^s| \right\},$$

where the a_j are real, contains no more than Kns points of Λ. It is not known whether this necessary condition is also sufficient. The triumph of the method is an alternative definition of Sidon sets, from which the theorem of Drury—the union of two Sidon sets is a Sidon set—follows immediately:

D2. Λ is a Sidon set if and only if $C_{as\,\Lambda} = A_\Lambda$.

This is due to D. Rider [52]. The characterizations given by Pisier [49, 50] and by Borgain [6] use the same approach.

The second method was introduced by Katznelson and Malliavin in 1966 [36] in connection with the "dichotomy problem": For a rather large class of random sets of integers, $\Lambda = \Lambda(\omega)$, either Λ is a Sidon set or only analytic functions operate on $\widehat{L^1}|_\Lambda$. Katznelson proved the following result in 1972 [35]: In the second case, Λ is dense in the Bohr group, that is, the Bohr compactification of $\mathbb{Z}$. Moreover, he indicated a way to construct Rudin's $\Lambda(p)$ sets by a random procedure. (Λ is $\Lambda(p)$ means that $\widehat{L^2}|\Lambda(= l^2(\Lambda)) = \widehat{L^p}|_\Lambda$.) This was later developed by Borgain in [7] to obtain $\Lambda(p)$ sets that are not $\Lambda(p+\varepsilon)$. Borgain's selector method consists in choosing Λ as the support of the random measure $\sum_{n\in\mathbb{Z}} X_n\delta_n$, where δ_n is the Dirac mass at n, and where the X_n are independent random variables with values 0 or 1 such that $EX_n = a_n$ for some given sequence a_n, $0 < a_n < 1$. When $a_n = O(1/\log|n|)$, $n \to \pm\infty$, Λ is a.s. a Sidon set. When $|a_n|\log|n|$ tends to ∞, there is a.s. a trigonometric series $\sum_{\lambda\in\Lambda} c_\lambda \sin\lambda t$ that is uniformly, but not absolutely, convergent. New developments are given in [45].

It is through definition D2 that Sidon sets are related to almost surely everywhere convergent (or equivalently, uniformly convergent) trigonometric series. The Zygmund sets (or sequences), which I shall define below, are related to almost surely somewhere convergent trigonometric series, which we have already considered. They originate from a theorem of Zygmund [57] and a theorem of Paley and Mary Weiss [56] on Hadamard real trigonometric series and Hadamard Taylor series with coefficients tending to zero. From now on, Λ will denote a set of positive integers, ordered increasingly.

D3. Λ is a Zygmund set whenever every real trigonometric series

$$\sum_{\lambda\in\Lambda} \mathrm{Re}(a_\lambda e^{2\pi i\lambda t})$$

with $a_\lambda = o(1)$, $\lambda\to\infty$, converges at some point $t\in\mathbb{T}$.

D4. Λ is a Zygmund$^+$ set whenever every Taylor series $\sum_{\lambda\in\Lambda} a_\lambda z^\lambda$ with $a_\lambda = o(1)$, $\lambda\to\infty$, converges at some z, $|z| = 1$.

Every Zygmund$^+$ set is a Zygmund set. It is not known if the converse is true. Necessary conditions for Λ to be a Zygmund set were given by Erdős [13] and by me [21], and they are close to the necessary conditions known for Sidon sets. Is every Zygmund set a Sidon set? Is every Sidon set in $\mathbb{N}$ a Zygmund set? These are old questions, hardly considered.

8. Kronecker, Helson, M, and Salem sets

We are now going to consider closed subsets of $\mathbb{T}$. I shall restrict myself to Kronecker sets, Helson sets, M sets, M_α sets with $0 \leqslant \alpha < 1$, and Salem sets. Here are the definitions ([29, 33]).

D5. E (a closed subset of $\mathbb{T}$) is a Kronecker set if each function of modulus 1 that is continuous on E is the uniform limit of some sequence of imaginary exponentials $\exp(2\pi i n_j t)$.

From now on I shall write $C(E)$ for the Banach space of continuous functions on E and $A(E)$ for the Banach space consisting of the restrictions to E of functions belonging to $A(\mathbb{T})$.

D6. E is a Helson set if $A(E) = C(E)$.

An equivalent definition is that the measure norm and the pseudomeasure norm are equivalent for all measures $\mu \in M(E)$, that is, measures supported by E. The first theorem about Helson sets, due to Helson, is that a Helson set carries no nonzero measure belonging to $M_0(\mathbb{T})$ [16]. A long standing question, whether a Helson set can carry a pseudofunction, was solved positively by T. Körner in 1972 [41]. The best proof uses a probability device of R. Kaufman [38].

D7. E is an M set if it carries a nonzero pseudofunction, or in our notation, if $PF(E) \neq \{0\}$.

An M set is also called a set of multiplicity because there are infinitely many trigonometric series that converge to 0 outside the set. The opposite is called a set of uniqueness. A Kronecker set is both a Helson set and a set of uniqueness.

D8. E is an M_0 set if it carries a nonzero measure μ such that $\hat{\mu}(n) = o(1)$ as $|n| \to \infty$.

Helson's theorem says that an M_0 set cannot be a Helson set; Körner's construction shows that an M set can be a Helson set.

D9. E is an M_α set if $M_\alpha(E) \neq \{0\}$, which means that E supports a probability measure μ such that $\hat{\mu}(n) = o(|n|^{-\alpha/2})$ as $|n| \to \infty$.

It follows from a theorem of Frostman that the Hausdorff dimension of an M_α set is $\geqslant \alpha$.

D10. E is a Salem set of dimension α if its Hausdorff dimension is α and if E is an M_β set for every $\beta < \alpha$.

9. Random thin sets

Probability methods have been used to obtain Salem sets in different ways. The problem of finding Salem sets of dimension α for each α between 0 and 1 was suggested by Beurling [33].

Salem's construction, apart from probability methods, uses Diophantine approximation, and it produces sets that have strong arithmetical properties, of the same type as the usual Cantor sets. This construction is described in [33].

Processes with independent increments, in particular Lévy processes and Brownian motion, provide Salem sets in a most natural way: The images of a fixed set of dimension $\alpha/2$ under Brownian motion is a.s. a Salem set [24]. There is an analogue of this result for Lévy flights [26, 30, 46]. The first result in this direction was suggested by B. Mandelbrot when he became interested in the "Lévy dust." Here is the result. Recall that any stationary increasing process with independent increments, a Lévy flight, is a mapping $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ that satisfies the following properties: For almost all ω, $X(t, \omega)$ is an increasing function of t, it is continuous to the right, and $X(0, \omega) = 0$ for all ω. Furthermore, for each choice of $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$, the random variables $X(t_{j+1}, \cdot) - X(t_j, \cdot)$, $j = 1, 2, \ldots, n-1$, are independent and their law depends only on $t_{j+1} - t_j$. The "ψ function" associated with this process is defined by the equation

$$Ee^{iuX(t)} = e^{-t\psi(u)}, \quad X(t) = X(t, \cdot),$$

and the law of the process depends only on ψ. Then the image of the Lebesgue measure on $[0, 1]$ under $X(t)$ is a random measure μ whose Fourier transform is

$$\hat{\mu}(u) = \int_0^1 e^{iuX(t)}\, dt.$$

Writing

$$h(u) = \inf_{|x| \geqslant u} \operatorname{Re} \psi(x),$$

one gets successively

$$E|\hat{\mu}(u)|^{2p} \leqslant \frac{(2p)!\, 2^p}{p!\, (h(u))^p},$$

and

$$\hat{\mu}(u) = O\left(\sqrt{\frac{\log u}{h(|u|)}}\right), \quad |u| \to \infty, \quad \text{a.s.}$$

The last steps are copied from Salem [33]. The definition of $h(u)$ as $\inf |\psi(x)|$ given in [30] and [26] is incorrect. The correction is due to A. Benchérif–Madani (see [5], p. 87). In particular, for stable Lévy processes of index α, $\psi(u) = cu^\alpha$, where c is complex, and the support of μ, transported to $\mathbb{T}$, is a Salem set a.s.

It can be expected that most "naturally occurring" random sets are Salem sets. In particular, level sets of fractional Brownian motions should be Salem sets. This is true for ordinary Brownian motion because the level set starting from 0 coincides with the closure of the range of a Lévy process of index $1/2$. The measure to be considered is now $\delta(X(\cdot))$. The method is at hand in the last chapter of [24], but it was never carried out.

Examples of sets where spectral synthesis fails on the line $\mathbb{R}$ were first given by Malliavin. His idea was to construct a function $f \in A(\mathbb{T})$ and a pseudomeasure defined formally as $\delta'(f)$, where δ' is the derivative of the Dirac measure. When $\langle \delta'(f), f \rangle \neq 0$, the set $f^{-1}(0)$ does not permit spectral synthesis. Level sets of specially constructed Gaussian processes $X(t)$ have this property: $\delta'(X(\cdot) - x)$ is a pseudomeasure that cannot be approximated in the weak topology of $PM(\mathbb{T})$ by measures carried within the support of $\delta'(X(\cdot) - x)$, with a positive probability that depends on x. It was noticed that, for these special Gaussian processes, this pseudomeasure belongs to $\mathcal{F}l^p$ whenever $p > 2$ [20]. (The statement in [20] is: Whatever $p > 2$, there exists a process.... It is easy to obtain: There exists a process such that, whatever $p > 2, \ldots$.) It is very likely that, for these Gaussian processes, the pseudomeasure belongs to $M_\alpha(\mathbb{T})$, whatever $0 < \alpha < 1/2$ (perhaps also for $\alpha = 1/2$), but it needs some work to prove this.

10. Generic thin sets

I shall be brief on Baire's methods—as introduced by Robert Kaufman [37], developed in a series of ways, and revived recently by Thomas Körner [42]—because the matter is treated in [28]. However, it is worth mentioning the power and versatility of Baire's methods for obtaining strange closed sets on $\mathbb{R}$ or $\mathbb{T}$. (Definitions D5 to D10 extend to $\mathbb{R}$, so I shall consider $\mathbb{R}$ instead of $\mathbb{T}$ from now on.)

The first idea is that, contrary to the smoothing effect of random processes, using Baire's theorem accentuates the wildest behavior of Fourier transforms: Probability provides M sets, and even M_α sets, as we have seen; Baire provides sets of uniqueness. Actually, Baire's theorem is a very good tool for obtaining Kronecker sets [22] or, in several dimensions, Helson curves and surfaces. That at least was my personal philosophy until 1993.

Then T. Körner discovered that his Helson–M sets (so strange!) and also Salem sets (so far from Kronecker!) are also generic if suitable Baire spaces are chosen [42]. His ideas for the construction of Salem sets are expounded in [28]. Probability does not disappear, but it is reduced to a technicality.

In [42], Körner called Baire's theorem "a profound triviality." Its proof is trivial, and its use by Kaufman and Körner is profound. The use of probability methods relies on long experience in analysis and on the stochastic processes that we have at hand. To apply Baire's theorem, the first point is to discover the right Baire space; the second step is to chose the right open sets. It is a good way both to render strange objects generic and to find new phenomena.

I wish to thank Robert Ryan for a careful reading of this paper.

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