How to Use the Fourier Transform in Asymptotic Analysis

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ABSTRACT. This introductory paper presents a method for the analysis of differential equations with polynomial coefficients which also provides a further insight into the Stokes Phenomenon. The method consists of a chain of steps based on the concept of the Stokes Structure and Fourier-like transforms adjusted to this Stokes Structure. Although the main object here is Bessel's equation our approach can be extended to more general matrix equations. It will be shown (i) how to derive the Stokes Structure directly from differential equations without any previous knowledge of Bessel or hypergeometric functions, (ii) how to adjust Fourier transforms to the Stokes Structure, (iii) how to answer questions on the interrelation between formal and actual solutions of Bessel's equation using Fourier Analysis, and finally (iv) how to evaluate the coefficients of the Stokes Structure, thus providing a new insight into the Stokes Phenomenon.

1. Introduction

In [4], [5] an approach for the study of a general class of matrix differential equations with polynomial coefficients was presented. However, this study does not cover many equations which require special attention. One such case is the classical Bessel's equation. It was explained in [3] how to derive properties of solutions of Bessel's equation from the Fourier-dual hypergeometric equations. In particular, it was shown how the monodromic properties of hypergeometric functions are transfered to solutions of Bessel's equation as algebraic relations.

The Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ of order ν (or Bessel functions of the third kind) are unique solutions of Bessel's equation

$$y'' + \frac{1}{z}y' + \left(1 - \frac{\nu^2}{z^2}\right)y = 0 \tag{1.1}$$

satisfying the Hankel inequalities (or expansions)

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-\nu\pi/2-\pi/4)}\,(1+o(1)) \tag{1.2}$$

$$H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z-\nu\pi/2-\pi/4)}\,(1+o(1)) \tag{1.3}$$

as $z \to +\infty$. They can be continued analytically as single-valued functions to the whole Riemann surface of log z : $0 < |z| < \infty,\ -\infty < \arg z < +\infty$.

The functions $P_1(z),\ P_2(z)$ defined by

$$H_\nu^{(1)}(z) \equiv \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-\nu\pi/2-\pi/4)}\,P_1(z) \tag{1.4}$$

$$H_\nu^{(2)}(z) \equiv \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z-\nu\pi/2-\pi/4)}\,P_2(z) \tag{1.5}$$

are known as the *phase amplitudes* of the Hankel functions $H_\nu^{(1)}(z),\ H_\nu^{(2)}(z)$ respectively. It follows that

$$P_1(z) = 1 + o(1),\ P_2(z) = 1 + o(1) \tag{1.6}$$

as $z \to +\infty$. They can also be extended as analytic single-valued functions to the whole Riemann surface of log z. Moreover, they satisfy respectively the following pair of differential equations

$$\mathcal{L}_1 P_1(z) = 0,\ \mathcal{L}_2 P_2(z) = 0 \tag{1.7}$$

with the pair of differential operators $\mathcal{L}_1,\ \mathcal{L}_2$ defined by

$$\mathcal{L}_1 = z^2 D_z^2 + 2iz^2 D_z - b \tag{1.8}$$

and

$$\mathcal{L}_2 = z^2 D_z^2 - 2iz^2 D_z - b \tag{1.9}$$

where $b = \nu^2 - 1/4$ and $D_z \stackrel{\text{def}}{=} \frac{d}{dz}$.

On the other hand there exists a unique pair of factorially divergent power series

$$\hat{P}_1(z) = 1 + \sum_{m=1}^{\infty} \frac{a_{1,m}}{z^m},\ \hat{P}_2(z) = 1 + \sum_{m=1}^{\infty} \frac{a_{2,m}}{z^m} \tag{1.10}$$

formally satisfying equations (1.7) respectively.

It is natural to introduce the Fourier-dual operators $\mathcal{L}_1^*,\ \mathcal{L}_2^*$ to $\mathcal{L}_1,\ \mathcal{L}_2$

$$\mathcal{L}_1^* \stackrel{\text{def}}{=} \xi\,(\xi - 2i)\,D_\xi^2 + 2\,(\xi - i)\,D_\xi - b \tag{1.11}$$

$$\mathcal{L}_2^* \stackrel{\text{def}}{=} \xi\,(\xi + 2i)\,D_\xi^2 - 2\,(\xi + i)\,D_\xi - b \tag{1.12}$$

where $D_\xi \stackrel{\text{def}}{=} \frac{d}{d\xi}$.

There exists a unique pair $F_1(\xi),\ F_2(\xi)$ of solutions of $\mathcal{L}_1^* F_1(\xi) = 0$, $\mathcal{L}_2^* F_2(\xi) = 0$ respectively, analytic at the singular point $\xi = 0$. This pair is nothing but the pair of Gauss hypergeometric functions

$$F_1(\xi) = F\left(\frac{1}{2} - \nu, \frac{1}{2} + \nu, 1, \xi/2i\right) \tag{1.13}$$

$$F_2(\xi) = F\left(\frac{1}{2} - \nu, \frac{1}{2} + \nu, 1, -\xi/2i\right). \tag{1.14}$$

It is not difficult to check that the formal power series $\hat{P}_1(z),\ \hat{P}_2(z)$ can be represented respectively as formal Laplace transforms of the formal hypergeometric series

$$\sum_{m=0}^{\infty} \frac{(\frac{1}{2}+\nu)_m(\frac{1}{2}-\nu)_m}{(2i)^m(1)_m m!}\xi^m \tag{1.15}$$

$$\sum_{m=0}^{\infty} (-1)^m \frac{(\frac{1}{2}+\nu)_m(\frac{1}{2}-\nu)_m}{(2i)^m(1)_m m!}\xi^m \tag{1.16}$$

where

$$(a)_m \stackrel{\text{def}}{=} a(a+1)\ldots(a+m-1) = \frac{\Gamma(a+m)}{\Gamma(a)}, \tag{1.17}$$

while the phase amplitudes $P_1(z),\ P_2(z)$ can be represented as classical Laplace transforms of (1.13), (1.14) respectively, see [3]. In other words, these formal series and the phase amplitudes are generated in the same manner by different branches of the same hypergeometric function.

Moreover, using (1.4), (1.5) we obtain the following integral representations of Hankel functions

$$H_\nu^{(1)}(z) = \left(\frac{2z}{\pi}\right)^{1/2} e^{i\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)} \int_0^{+\infty} e^{-z\xi} F\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu, 1, \frac{\xi}{2i}\right) d\xi \tag{1.18}$$

$$H_\nu^{(2)}(z) = \left(\frac{2z}{\pi}\right)^{1/2} e^{-i\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)} \int_0^{+\infty} e^{-z\xi} F\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu, 1, -\frac{\xi}{2i}\right) d\xi, \tag{1.19}$$

which, upon using the monodromic properties of hypergeometric functions, yield the following monodromic relation, see [3]

$$\begin{pmatrix} 1 & 0 \\ -T_2 e^{2iz} & 1 \end{pmatrix} \begin{pmatrix} P_1\left(ze^{2\pi i}\right) \\ P_2\left(ze^{2\pi i}\right) \end{pmatrix} = \begin{pmatrix} 1 & T_1 e^{-2iz} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P_1(z) \\ P_2(z) \end{pmatrix} \tag{1.20}$$

where T_1, T_2 are complex constants.

This relation suggests an algebraic structure for the phase amplitudes $P_1(z)$, $P_2(z)$ on the Riemann surface of log z, which will form the basis of our present investigation. The principal idea of this paper is to apply Fourier transforms to this algebraic structure rather than to the original differential equation. It should be noted in fact that our approach presented in Sections 2, 6, 7, 8, 9, 10 to follow does not depend on the original differential equation.

2. The Stokes Structure $\mathfrak{S}$

DEFINITION *A pair of functions* $P_1(z)$, $P_2(z)$

(i) *analytic on the Riemann surface of* $\log z$ *with at most exponential growth at* $z = \infty$ *in every sector* $S_{\alpha,\beta} = \{z : -\infty < \alpha < \arg z < \beta < +\infty\}$

(ii) *satisfying inequalities*

$$P_1(z) = 1 + o(1), \; z \to \infty, \; z \in S_c(1) \tag{2.1}$$

$$P_2(z) = 1 + o(1), \; z \to \infty, \; z \in S_c(2) \tag{2.2}$$

in the closed subsectors

$$S_c(1) \subset S(1) \overset{\text{def}}{=} \{z : -\pi < \arg z < 2\pi, \; 0 < |z| < \infty\} \tag{2.3}$$

$$S_c(2) \subset S(2) \overset{\text{def}}{=} \{z : -2\pi < \arg z < \pi, \; 0 < |z| < \infty\} \tag{2.4}$$

(iii) *satisfying the monodromic relation (1.20) with complex constants* T_1, T_2 *written as*

$$P_1(ze^{2\pi i}) = P_1(z) + T_1 P_2(z) e^{-2iz} \tag{2.5}$$

$$P_2(ze^{2\pi i}) = P_2(z) + T_2 P_1(ze^{2\pi i}) e^{2iz} \tag{2.6}$$

are the elements of the Stokes Structure

$$\mathfrak{S} = \{P_1(z), P_2(z)\}. \tag{2.7}$$

3. From Differential Equation to $\mathfrak{S}$

This technique does not require any previous knowledge or properties of the solutions of (1.1) nor of the hypergeometric functions.

DEFINITION *The rays*

$$l = \{z : \ Re(iz) = 0\} \tag{3.1}$$

are called $separationrays$ *for (1.1).*

Let us look for solutions y_1, y_2 of (1.1)

$$y_1(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-\nu\pi/2-\pi/4)} \, P_1(z) \tag{3.2}$$

$$y_2(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z-\nu\pi/2-\pi/4)} \, P_2(z) \tag{3.3}$$

such that

$$P_1(z) = 1 + o(1), \ P_2(z) = 1 + o(1) \tag{3.4}$$

as $z \to \infty$ along a separation ray l on the Riemann surface of log z.

In terms of $P_1(z)$, $P_2(z)$ the differential equations (1.7) together with conditions (3.4) can be equivalently rewritten respectively as

$$P_1(z) = 1 - \frac{b}{2i}\int_z^{\infty_l} \frac{P_1(w)}{w^2}dw \, + \frac{b}{2i}\int_0^{\infty_l} e^{2iw}\frac{P_1(w+z)}{(w+z)^2}dw \tag{3.5}$$

$$P_2(z) = 1 + \frac{b}{2i}\int_z^{\infty_l} \frac{P_2(w)}{w^2}dw \, - \frac{b}{2i}\int_0^{\infty_l} e^{-2iw}\frac{P_2(w+z)}{(w+z)^2}dw \tag{3.6}$$

with $\infty_l = \infty \cdot e^{i\,\arg l}$.

The integral equations (3.5), (3.6) can be analyzed using successive iterations to construct the unique solutions $P_1(z)\,, P_2(z)$ satisfying inequalities (3.4) respectively, see, for example [2]. Further analysis of these integral equations for a specially chosen l shows that $P_1\,(z)\,, P_2\,(z)$ form in fact the Stokes Structure $\mathfrak{S}$ defined above by (2.7).

4. Formal and Actual Solutions

Choosing the separation ray arg $z = 0$ as the paths of integration in (3.5), (3.6) respectively to construct the solutions $P_1(z),\ P_2(z)$ and using the uniqueness of this pair and that of $H_\nu^{(1)}(z),\ H_\nu^{(2)}(z)$ also yield

(4.1) $$H_\nu^{(1)}(z) \equiv y_1(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-\nu\pi/2-\pi/4)}\, P_1(z)$$

(4.2) $$H_\nu^{(2)}(z) \equiv y_2(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z-\nu\pi/2-\pi/4)}\, P_2(z)$$

which are identical to (1.4), (1.5). Thus the solutions of (3.5), (3.6) for this chosen separation ray are nothing but the *phase amplitudes* $P_1(z),\ P_2(z)$ of the Hankel functions $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ respectively.

Another pair of linearly independent solutions of (1.1) is

(4.3) $$\hat{y}_1(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-\nu\pi/2-\pi/4)}\, \hat{P}_1(z)$$

(4.4) $$\hat{y}_2(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z-\nu\pi/2-\pi/4)}\, \hat{P}_2(z)$$

where

(4.5) $$\hat{P}_1(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}+\nu)_m(\frac{1}{2}-\nu)_m}{(2i)^m(1)_m}\frac{1}{z^m}$$

(4.6) $$\hat{P}_2(z) = \sum_{m=0}^{\infty} (-1)^m \frac{(\frac{1}{2}+\nu)_m(\frac{1}{2}-\nu)_m}{(2i)^m(1)_m}\frac{1}{z^m}.$$

Formal substitution of these solutions into (1.1) yield, after canceling the exponentials, power series in z^{-1} with zero coefficients. However, the above power series are clearly factorially divergent for any z if ν is not a half integer. Thus, these solutions can be regarded as *formal solutions* as opposed to *actual solutions.*

Three natural questions arise immediately:

(1) how to relate the pair of formal solutions one to another,

(2) how to relate the pair of formal solutions to actual solutions $H_\nu^{(1)}(z),\ H_\nu^{(2)}(z)$,

(3) how to decode properly the symbol $o\,(1)$ in the expansions above.

Stokes (1857) was the first one to formulate and answer the first two questions for Airy's differential equation $y'' - zy = 0$ related to Bessel's equation for $\nu = \frac{1}{3}$. To answer the third question, Poincaré (1886) considered formal solutions as asymptotic representations of actual solutions. However, as discovered a century later, see [1], this approach is not satisfactory since it does not answer question (1) altogether, only answers partially question (2), and does not provide sufficient information about the remainder.

5. The Stokes Phenomenon

Using (4.1), (4.2) the monodromic relations (2.5), (2.6) can be rewritten in terms of $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ as

$$H_\nu^{(1)}\left(ze^{2\pi i}\right) = -H_\nu^{(1)}(z) + ie^{-i\nu\pi}T_1 H_\nu^{(2)}(z) \tag{5.1}$$

$$H_\nu^{(2)}\left(ze^{2\pi i}\right) = -H_\nu^{(2)}(z) + ie^{i\nu\pi}T_2 H_\nu^{(1)}\left(ze^{2\pi i}\right). \tag{5.2}$$

These, in turn, yield extended Hankel expansions valid outside the sectors in (2.3), (2.4).

All these Hankel expansions are of the form

$$z^{-1/2}\left(A(\nu)e^{iz} + B(\nu)e^{-iz}\right). \tag{5.3}$$

Again, Stokes (1857) was the first to discover that the constants $A(\nu)$ and $B(\nu)$ are discontinuous as arg z changes continuously when crossing the separation rays. The existence of such discontinuities is called the *Stokes Phenomenon* and the corresponding values of the jumps in $A(\nu)$, $B(\nu)$ can be expressed in terms of *connection coefficients* T_1, T_2 very important in many applications. A modern insight into the Stokes Phenomenon can be found in [1].

A fourth question then arises:

(4) how to evaluate the connection coefficients T_1, T_2.

Starting with the Stokes Structure we will present a technique that answers questions (1)-(3). The culmination of our approach will be to answer question (4), obtaining explicit expressions for the connection coefficients independently of any knowledge of the actual solutions of the differential equation.

6. Fourier-Like Transforms Adjusted to $\mathfrak{S}$

Let $P_1(z)$, $P_2(z)$ be elements of $\mathfrak{S}$ with (unknown) T_1, T_2 in its monodromic relations (2.5), (2.6) and $S_c(1) \subset S(1)$, $S_c(2) \subset S(2)$ a pair of closed subsectors with angles greater than π.

Let

$$H(z) = a_0 z(1 + o(1)), \ z \to \infty \tag{6.1}$$

be analytic on the Riemann surface of log z, and

$$C(1/z) = c_0 + c_1 z + \ldots \tag{6.2}$$

an entire function with complex $a_0 \neq 0;\ c_0,\ c_1, \ldots$.

We define the general Fourier-like transforms of $P_j(z),\ j=1,2$ as

$$F_j(\xi) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{\gamma(j)} e^{H(z\xi)}C(z)\,P_j(z)\,dz/z,\ j=1,2 \tag{6.3}$$

with paths of integration $\gamma(1)$, $\gamma(2)$ as boundaries of $S_c(1)$, $S_c(2)$ respectively, oriented so that $S_c(j)$ are to the right of $\gamma(j)$.

7. Main Result

THEOREM 1. *Let $P_1(z)$, $P_2(z)$ be the elements of the Stokes Structure $\mathfrak{S}$.*
Then for each $j=1,2$ in the dual complex $\xi-$plane

(i) *there exists a ray l_j emanating from the origin such that $F_j(\xi)$ is continuous for $\xi \in l_j$, and $F_j(\xi)$ can be continued analytically to some open sector containing the ray l_j;*
(ii) *there exists a neighborhood of the origin such that $F_j(\xi)$ can be further continued analytically to this neighborhood as a single-valued function;*
(iii) *moreover, $F_j(\xi)$ can be continued analytically to the whole $\xi-$plane along every path not crossing the point*

$$\xi_0 \equiv \xi_{0,j} = \frac{2i}{a_0}(-1)^{j-1}. \tag{7.1}$$

8. From $\mathfrak{S}$ to Formal Power Series

Consider the special cases of Fourier-like transforms (6.3) for $H(z) = z$, $C(z) = 1$. These are nothing but the Borel transforms of $P_j(z)$

$$F_j^{(0)}(\xi) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{\gamma(j)} e^{z\xi}P_j(z)\,dz/z,\ j=1,2. \tag{8.1}$$

Their inversion formulae are nothing but the Laplace transforms of $F_j^{(0)}(\xi)$

$$P_j(z) = z\int_{l_j} e^{-z\xi}F_j^{(0)}(\xi)\,d\xi,\ j=1,2. \tag{8.2}$$

Due to (i), (ii) of Theorem 1 the integrals (8.1) are absolutely convergent for $\xi \in l_j$ and $F_j^{(0)}(\xi)$ can be represented by their Taylor series, which can be regarded as formal power series in ξ

$$\hat{F}_j^{(0)}(\xi) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} f_{j,k}^{(0)}\xi^k. \tag{8.3}$$

Substituting $\hat{F}_j^{(0)}(\xi)$ for $F_j^{(0)}(\xi)$ in (8.2) and writing

$$a_{j,k} \stackrel{\text{def}}{=} k!f_{j,k}^{(0)} \tag{8.4}$$

yield

$$\hat{P}_j^{(0)}(z) \overset{\text{fps}}{=} z \int_{l_j} e^{-z\xi} \hat{F}_j^{(0)}(\xi)\, d\xi = \sum_{k=0}^{\infty} \frac{a_{j,k}}{z^k},\ j = 1, 2. \tag{8.5}$$

The symbol fps means that (8.5) should be perceived on the level of formal power series.

9. Formal Series as Strong Expansions

Although for an element P_j of the Stokes Structure (2.7)

$$\lim_{z\to\infty\, z\in S_c(j)} P_j(z) = 1, \tag{9.1}$$

it is not at all obvious that the Stokes Structure guarantees the next limits

$$\lim_{z\to\infty\, z\in S_c(j)} (P_j(z) - 1)z. \tag{9.2}$$

However, the formal series

$$\sum_{k=0}^{\infty} \frac{a_{1,k}}{z^k},\ \sum_{k=0}^{\infty} \frac{a_{2,k}}{z^k} \tag{9.3}$$

are Poincaré asymptotic expansions for $P_1(z)$, $P_2(z)$ in sectors $S(1)$, $S(2)$ respectively. This means that for any subsector $S_c(j)$ of $S(j)$ and for $z \in S_c(j)$ there exists $M_N > 0$ such that the following estimates are valid for $N = 1, 2, \ldots$

$$\left| P_j(z) - \sum_{k=0}^{N-1} \frac{a_{j,k}}{z^k} \right| < \frac{M_N}{|z|^N}. \tag{9.4}$$

It should be noted, however, that these approximations are too rough to provide real information about the behavior of the remainders

$$P_j(z) - \sum_{k=0}^{N-1} \frac{a_{j,k}}{z^k} \tag{9.5}$$

since we don't know how M depends on N.

In fact the formal series (9.3) are much better and more precise asymptotic expansions for $P_j(z)$ than the Poincaré expansions.

THEOREM 2. *For any subsector $S_c(j)$ of $S(j)$ and for $z \in S_c(j)$ there exists $a > 0$ depending only on $S_c(j)$ such that the following estimates are valid for $N = 1, 2, \ldots$*

$$\left| P_j(z) - \sum_{k=0}^{N-1} \frac{a_{j,k}}{z^k} \right| < \frac{M a^N N!}{|z|^N}. \tag{9.6}$$

These expansions are known as *strong asymptotic expansions*, see [7], [6]. In contrast to Poincaré expansions they have the following uniqueness property:

WATSON'S THEOREM. Watson's Theorem *If* $P_1(z)$, $P_2(z)$ *are analytic functions in a sector* S *with its angle not less than* π, *and* $\sum_{k=0}^{\infty} \frac{a_k}{z^k}$ *is their strong asymptotic expansion in* S, *then* $P_1(z) \equiv P_2(z)$.

The inequalities (9.6) answer our question (3).

10. Power Series Representation of $F_j(\xi)$

Now that we have $\sum_{k=0}^{\infty} \frac{a_{j,k}}{z^k}$ it is natural to formally substitute these for $P_j(z)$ into the general Fourier-like transforms (6.3) to yield the formal Fourier-like transforms

$$\hat{F}_j(\xi) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{\gamma(j)} e^{H(z\xi)} C(z) \hat{P}_j^{(0)}(z)\, dz/z \tag{10.1}$$

resulting in the power series in ξ

$$\hat{F}_j(\xi) = \sum_{k=0}^{\infty} f_{j,k} \xi^k \equiv \sum_{k=0}^{\infty} \xi^k s_k \left(\sum_{m=0}^{k} a_{j,m} c_{k-m} \right) \tag{10.2}$$

with

$$s_k = \frac{1}{2\pi i} \int_{\gamma^*(j)} e^{H(z)} \frac{1}{z^{k+1}} dz, k = 0, 1, \dots \tag{10.3}$$

and

$$\gamma^*(j) = \gamma(j)\, e^{i \arg \xi}. \tag{10.4}$$

THEOREM 3. *The power series* $\hat{F}_j(\xi)$ *are absolutely convergent and thus represent the analytic functions*

$$\widetilde{F}_j(\xi) = \sum_{k=0}^{\infty} \xi^k \left(s_k \left(\sum_{m=0}^{k} a_{j,m} c_{k-m} \right) \right) \tag{10.5}$$

inside the circle of radius $\frac{2}{|a_0|}$ *with its center at* $\xi = 0$. *Moreover, if* $\xi \in l_j$ *and* $|\xi| < \frac{2}{|a_0|}$ *then*

$$\widetilde{F}_j(\xi) \equiv F_j(\xi). \tag{10.6}$$

Thus, the Fourier-like transforms $F_j(\xi)$ can be represented both by the integral transforms (6.3) and by the convergent Taylor series (10.5) for $\xi \in l_j$, $|\xi| < |\xi_0|$, where l_j and ξ_0 are defined in Theorem 1 (i) and (iii), (7.1) respectively.

11. Evaluation of Borel Transforms

Now let us return to Bessel's equation (1.1) and remember that the elements $P_1(z)$, $P_2(z)$ of the Stokes Structure (2.7) are the phase amplitudes of the Hankel functions $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$.

It follows from Theorem 2 that in particular the formal series (9.3) are Poincaré asymptotic expansions of $P_1(z)$, $P_2(z)$. On the other hand, one can derive from integral equations (3.5), (3.6) that the formal power series (4.5), (4.6) are also Poincaré asymptotic expansions of $P_1(z)$, $P_2(z)$. It should be noted, however, that it is a hard problem to derive from integral equations (3.5), (3.6) that the formal power series (4.5), (4.6) are strong asymptotic expansions for $P_1(z)$, $P_2(z)$.

The uniqueness property of Poincaré asymptotic expansions yields

$$a_{1,k} = \frac{(\frac{1}{2}+\nu)_k(\frac{1}{2}-\nu)_k}{(2i)^k(1)_k} \tag{11.1}$$

$$a_{2,k} = (-1)^k\frac{(\frac{1}{2}+\nu)_k(\frac{1}{2}-\nu)_k}{(2i)^k(1)_k} \tag{11.2}$$

that is

$$\hat{P}_j(z) \equiv \hat{P}_j^{(0)}(z) \tag{11.3}$$

and the left-, right-hand sides of (11.3) are defined by (8.5) and by (4.5), (4.6) respectively.

It is worth noting that (11.3) is in fact the converse of an important principle that was named in [3] as the *Principle of Functional Closure*: *If a formal series satisfying a differential-difference-algebraic relation can be summed to an analytic function in a region of the complex plane, then this function satisfies exactly the same relation in this region.*

It follows from (8.4) and (8.3) that

$$\hat{F}_j^{(0)}(\xi) = \hat{F}\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu, 1, \pm\frac{\xi}{2i}\right) \tag{11.4}$$

where $\hat{F}\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu, 1, \pm\frac{\xi}{2i}\right)$ are power series expansions in ξ of Gauss' hypergeometric function
$F\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu, 1, \pm\frac{\xi}{2i}\right)$, respectively.

12. Interrelation between Solutions

It follows from (11.4) and (8.1) that the Borel transforms of the phase amplitudes of the Hankel functions are in fact the hypergeometric functions, while the formal Borel transforms of the formal power series are the corresponding (formal) hypergeometric series.

The following relations are valid

$$F\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu,1,\frac{\xi}{2i}\right)=\frac{1}{2\pi i}\int_{\gamma(1)}e^{\xi z}P_1\left(z\right)dz/z \tag{12.1}$$

$$F\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu,1,-\frac{\xi}{2i}\right)=\frac{1}{2\pi i}\int_{\gamma(2)}e^{\xi z}P_2\left(z\right)dz/z \tag{12.2}$$

$$\hat{F}\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu,1,\frac{\xi}{2i}\right)\overset{\text{def}}{=}\frac{1}{2\pi i}\int_{\gamma(1)}e^{\xi z}\hat{P}_1\left(z\right)dz/z \tag{12.3}$$

$$\hat{F}\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu,1,-\frac{\xi}{2i}\right)\overset{\text{def}}{=}\frac{1}{2\pi i}\int_{\gamma(2)}e^{\xi z}\hat{P}_2\left(z\right)dz/z. \tag{12.4}$$

The representations (12.1)-(12.4) together with their respective inversion formulae

$$P_1\left(z\right)=z\int_0^\infty e^{-z\xi}F\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu,1,\frac{\xi}{2i}\right)d\xi \tag{12.5}$$

$$P_2\left(z\right)=z\int_0^\infty e^{-z\xi}F\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu,1,-\frac{\xi}{2i}\right)d\xi \tag{12.6}$$

$$\hat{P}_1\left(z\right)\overset{\text{def}}{=}z\int_0^\infty e^{-z\xi}\hat{F}\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu,1,\frac{\xi}{2i}\right)d\xi \tag{12.7}$$

$$\hat{P}_2\left(z\right)\overset{\text{def}}{=}z\int_0^\infty e^{-z\xi}\hat{F}\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu,1,-\frac{\xi}{2i}\right)d\xi \tag{12.8}$$

reveal the following one-to-one correspondences (denoted by the symbol $\leftrightarrow$) below

$$\begin{aligned}&\hat{P}_j\left(z\right)\leftrightarrow\hat{F}_j\left(\xi\right)\equiv\hat{F}\left(\tfrac{1}{2}+\nu,\tfrac{1}{2}-\nu,1,\pm\tfrac{\xi}{2i}\right)\leftrightarrow\\&\leftrightarrow F\left(\tfrac{1}{2}+\nu,\tfrac{1}{2}-\nu,1,\pm\tfrac{\xi}{2i}\right)\leftrightarrow P_j\left(z\right).\end{aligned} \tag{12.9}$$

These interrelations show that both formal series $\hat{P}_1\left(z\right)$, $\hat{P}_2\left(z\right)$ and actual functions $P_1\left(z\right)$, $P_2\left(z\right)$, are generated in the same manner by different branches of the same hypergeometric function, thus answering questions (1) and (2).

REMARK

Formulae (12.5), (12.6) together with (4.1), (4.2) yield again the integral representations (1.18), (1.19) for Hankel Functions. It is curious that we could not find these representations, the most principal in our context, in the classical literature on Bessel functions. In the literature, the Hankel expansions are commonly derived from the representations

$$H_\nu^{(1)}\left(z\right)=\frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{\pi^{3/2}i}\int_{\gamma_1}e^{izt}\left(t^2-1\right)^{\nu-\frac{1}{2}}dt \tag{12.10}$$

(12.11) $$H_\nu^{(2)}(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{\pi^{3/2} i} \int_{\gamma_2} e^{-izt}\left(t^2-1\right)^{\nu-\frac{1}{2}} dt$$

with $\gamma_1,\ \gamma_2$ *simple loops bypassing* $t=\pm 1$ *but not enclosing* $t=\mp 1$*, respectively,* $|\arg z| < \frac{\pi}{2}$*, and* $\nu \neq \frac{1}{2}, \frac{3}{2}, \ldots$.

These are derived by reducing Bessel's equation to

(12.12) $$zw'' + (2\nu+1)\, w' + zw = 0$$

for the variable $w = z^{-\nu} y$*, and then applying the Laplace transform to this special equation with linear coefficients. Unfortunately, this approach is generally not possible for other differential equations.*

13. The Connection Coefficients

Consider the Fourier-like transforms $F_1(\xi),\ F_2(\xi)$ of $P_1(z),\ P_2(z)$ defined by (6.3) with

(13.1) $$H(z) = 2iz,\ C(z) = -\frac{b\pi}{z},\ b = \nu^2 - \frac{1}{4}.$$

(13.2) $$F_j(\xi) \stackrel{\text{def}}{=} -\frac{b}{2i}\int_{\gamma(j)} e^{2i\xi z}\frac{1}{z}P_j(z)\, dz/z,\ j=1,2.$$

THEOREM 4. *Let* $P_1(z),\ P_2(z)$ *be the phase amplitudes of* $H_\nu^{(1)}(z)$*,* $H_\nu{}^{(2)}(z)$ *with Fourier-like transforms* $F_1(\xi),\ F_2(\xi)$ *defined by (13.2). Then*

(i) *the only finite singular point of both analytic functions* $F_1(\xi)\ , -F_2(-\xi)$ *is* $\xi = 1$

(ii) *the limiting values of* $F_1(\xi),\ -F_2(-\xi)$ *at* $\xi = 1$ *exist and are equal to connection coefficients*

(13.3) $$\lim_{\xi\to 1} F_1(\xi) = T_1,\ \lim_{\xi\to 1}\left(-F_2(-\xi)\right) = T_2$$

(iii)

(13.4) $$T_j = (-1)^j\, \frac{b}{2i}\int_{\gamma^*(j)} e^{(-1)^{j-1}2iz}\frac{1}{z}P_j(z)\, dz/z,\ j=1,2$$

where $\gamma^*(j)$ *are obtained by rotating* $\gamma(j)$ *into positions where functions* $e^{(-1)^{j-1}2iz}$ *are decreasing for* $z \in \gamma^*(j),\ j=1,2$ *respectively.*

(iv) *Moreover, let* $f_{j,k}$ *be coefficients of power series in (10.2) for* $H(z)$ *and* $C(z)$ *given by (13.1). Then*

(13.5) $$T_1 = \lim_{\xi\to 1-0}\left(\sum_{k=0}^{\infty} f_{1,k}\xi^k\right)$$

$$T_2 = \lim_{\xi \to 1-0} \left(\sum_{k=0}^{\infty} (-1)^{k+1} f_{2,k} \xi^k \right). \tag{13.6}$$

It follows from Theorems 1 and 2 that for $|\xi| < 1$

$$\begin{aligned} F_1(\xi) &= -2\pi bi \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \frac{\left(\frac{1}{2}-\nu\right)_m \left(\frac{1}{2}+\nu\right)_m}{m!} \xi^m \\ F_2(\xi) &= 2\pi bi \sum_{m=0}^{\infty} \frac{(-1)^k}{(m+1)!} \frac{\left(\frac{1}{2}-\nu\right)_m \left(\frac{1}{2}+\nu\right)_m}{m!} \xi^m \end{aligned} \tag{13.7}$$

hence

$$F_1(\xi) = -F_2(-\xi) = -2\pi ibF\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu, 2, \xi\right). \tag{13.8}$$

Substituting $\xi = 1$ yields

$$T_j = -2\pi biF\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu, 2, 1\right), \; j = 1, 2 \tag{13.9}$$

which, using Gauss' formula, reduces to

$$T_j = \frac{-2\pi bi}{\Gamma\left(1+\frac{1}{2}+\nu\right)\Gamma\left(1+\frac{1}{2}-\nu\right)}, \; j = 1, 2, \tag{13.10}$$

and finally

$$T_j = 2i\cos\pi\nu, \; j = 1, 2. \tag{13.11}$$

It is worth noting that generally it is impossible to express T_j in terms of known fuctions.

Their integral representation should be used to evaluate them asymptotically for extremal values of parameters of the differential equation.

Their Taylor series representation should be used for their numerical evaluation.

14. Conclusions

We have shown that the Stokes Structure $\mathfrak{S}$ is of fundamental importance. Starting with Bessel's equation (1.1) we derived $\mathfrak{S}$ and introduced and studied Fourier-like transforms adjusted to $\mathfrak{S}$. These yielded formal power series that are in fact formal solutions of (1.7). Furthermore, as shown by (12.9) *the phase amplitudes and their respective formal series are generated in the same manner by different branches of the same hypergeometric function*. These provide the basis for a systematic chain of steps to answer questions (1), (2), (3), (4) above, and an approach which can be extended to matrix equations with many applications.

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