ON DEVELOPMENT OF DIFFERENCE METHODS FOR SOME CLASSICAL AND NON-CLASSICAL INITIAL-BOUNDARY VALUE PROBLEMS STATED FOR PLURI-PARABOLIC EQUATIONS

Ekaterine Gordeziani
Tbilisi State University, Georgia

ABSTRACT. *We describe research on some classical and non-classical initial-boundary value problems stated for pluri-parabolic equations, particularly the construction and investigation of difference algorithms for their resolution. We obtain uniqueness results, demonstrate an iteration process which reduces the solution of the non-local problem to that of the classical one and which converges geometrically, and build averaged decomposition algorithms of parallel count for pluri-parabolic equations.*

PREFACE. The work is devoted to the investigation and numerical realization problems of some mathematical models describing various phenomena. Considered non-classical mathematical models represent classical and non-local initial-boundary value problems stated for pluri-parabolic equations. Such problems arise during the investigation of most difficult and important problems of physics, technique and ecology and various other branches of science. The model considered hereby mainly concerns the description and analysis of diffusion and displacements of mixtures, particularly pollutions, in the rivers.

Experiments for research of mentioned processes are very expensive and, in some cases, even impossible. Application of mathematical modeling, numerical analysis and computation technologies through creating virtual images on computers is cost effective and sometimes the only way of studying these phenomena.

To the theoretical investigation of mentioned mathematical models are dedicated the researches of such famous mathematicians as J.-L. Lions, S.V. Vladimirov, A. Bousiani, A.A. Samarskii, A.V. Bitsadze, A.M. Il'in, M.A. Sapagovas, B.P. Paniakh, etc.

Thus, for the development and investigation of ecological problems application of mathematical methods and information technologies is the one of the most important means of research.

1^0. STATEMENT OF THE PROBLEM. There is stated the following problem: there is searched the function $u(x,t)\in C^{2,1}(D)\cap C^{1,0}(\overline{D})$, satisfying equation

$$Lu(x,t)=f(x,t),\quad (x,t)\in D, \tag{1.1}$$

and the initial and initial-boundary conditions,

$$\left.\begin{array}{l} u(x,0,t_2,...,t_m)=\varphi_{11}(x,t_2,...,t_m),\ x\in\overline{G},\ 0\le t_i\le T_i, i=\overline{2,m},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u(x,t_1,...,t_{m-1},0)=\varphi_{1m}(x,t_1,...,t_{m-1}),\ x\in\overline{G},\ 0\le t_i\le T_i, i=\overline{1,m-1}, \end{array}\right\} \tag{1.2}$$

$$u(x,t)=\varphi_2(x,t),\quad x\in\Gamma^+,\quad 0\le t_i\le T_i,\quad i=\overline{1,m}, \tag{1.3}$$

$$\beta\frac{\partial u(x,t)}{\partial\nu}+\alpha u(x,t)=\sum_{i=0}^{P}\alpha_i u(x_{\Gamma_i},t)+\varphi(x,t), \tag{1.4}$$

$$x\in\Gamma^-,\quad 0\le t_i\le T_i,\ i=\overline{1,m},\ x_{\Gamma_i}\in\Gamma_i,\ i=\overline{0,P},$$

where

$$L\equiv\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i(x,t)\frac{\partial}{\partial x_i}+c(x,t)-\sum_{i=1}^{m}\frac{\partial}{\partial t_i};$$

in (1.2)-(1.4) conditions there is assumed that conditions of compatibility are fulfilled; α, β, α_i $\left(i=\overline{0,P}\right)$ are given constants; φ_{1i} $\left(i=\overline{1,m}\right)$, $\varphi_2(x,t)$, $\varphi(x,t)$ and $f(x,t)$ are prescribed, sufficiently smooth functions defined on the corresponding definition areas; $\frac{\partial u}{\partial \nu}=\frac{\partial u}{\partial x_1}\cos(Ox_1,\vec{\nu})+\cdots+\frac{\partial u}{\partial x_n}\cos(Ox_n,\vec{\nu})$, $\vec{\nu}$ is the normal of Γ boundary, $(Ox_i,\vec{\nu})$ $\left(i=\overline{1,n}\right)$ are the angles between Ox_i axis and $\vec{\nu}$ normal vector; D – $(n+m)$-dimensional area in R^{n+m} space, $D=G\times\Omega$, $(x,t)=(x_1,...,x_n,t_1,...,t_m)\in D$; $G\subset R^n$, Γ is sufficiently smooth boundary of $\overline{G}=G\times\Gamma$; in addition $\Gamma=\Gamma^+\cup\Gamma^-$; in the G there are given Γ^i $(i=0,P)$ curves crossing the G without touching Γ^- and there exists diffeomorphism $I_i(\cdot)$ between Γ^i curves and Γ^- boundary, $I_i(\Gamma^i)=\Gamma$ $(i=0,P)$; $\Omega=(0,T_1)\times\cdots\times(0,T_m)$, where T_i $(i=0,m)$ are known constants. There is assumed that the following conditions are satisfied:

(A) for any $(x,t)\in D$ point and real vector $\zeta=(\zeta_1,...,\zeta_n)\neq 0$ the inequality $\underline{\alpha}\sum_{i=1}^{n}\zeta_i^2\le\sum_{i,j=1}^{n}a_{ij}\zeta_i\zeta_j\le\overline{\alpha}\sum_{i=1}^{n}\zeta_i^2$ is true, $\overline{\alpha}$, $\underline{\alpha}$ are given positive numbers;

(B) the coefficients of L operator are continuous functions in $\overline{D}$;

(C) $c(x,t)\le 0$ in $\overline{D}$.

For the pluri-parabolic operator the principle of maximum, which represents the analogue of well-known principle of maximum for the parabolic operator is proved.

2^0. UNIQUENESS OF SOLUTION. There is investigated the question of uniqueness for the problem stated in the previous paragraph. Using the principle of maximum the following theorem is proved:

Theorem 2.1. *If $\beta=0$, $\alpha\neq 0$, $\left|\sum_{i=0}^{m}\alpha_i\right|<|\alpha|$ in (1.4) condition and there exists solution of the problem (1.1)-(1.4), then the solution is unique.*

From problem (1.1)-(1.4) under the assumptions, that $G=(0,l_1)\times\cdots\times(0,l_n)$, $l_i=const$, $i=\overline{1,n}$ and $L\equiv\sum_{i=1}^{n}a_{ii}(x,t)\frac{\partial^2}{\partial x_i^2}+c(x,t)-\sum_{i=1}^{m}\frac{\partial}{\partial t_i}$, in addition $\alpha>0$, $\beta>0$, $a_{ii}\ge a_0=const>0$, $\left(i=\overline{1,n}\right)$, $a_{ii}=a_{ii}(x_1,...,x_{i-1},x_{i+1},...,x_n,t)$, the following problem is obtained:

$$\sum_{i=1}^{n}a_{ii}(x,t)\frac{\partial^2 u}{\partial x_i^2}+c(x,t)u-\sum_{i=1}^{m}\frac{\partial u}{\partial t_i}=f(x,t),\ (x,t)\in D, \tag{2.1}$$

$$\left.\begin{array}{l} u(x,0,...,t_m)=\varphi_{11}(x,t_2,...,t_m),\ x\in\overline{G}, 0\le t_i\le T_i, i=\overline{2,m},\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ u(x,t_1,...,t_{m-1},0)=\varphi_{1m}(x,t_1,...,t_{m-1}),\ x\in\overline{G}, 0\le t_i\le T_i, i=\overline{1,m-1}, \end{array}\right\} \tag{2.2}$$

$$\left.\begin{array}{l} u(0,...,x_n,t)=\varphi_{21}(x_2,...,x_n,t),\ t\in\overline{\Omega},\ 0\le x_i\le l_i, i=\overline{2,n},\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ u(x_1,...,x_{n-1},0,t)=\varphi_{2n}(x_1,...,x_{n-1},t),\ t\in\overline{\Omega},\ 0\le x_i\le l_i, i=\overline{1,n-1}, \end{array}\right\} \tag{2.3}$$

$$\left.\begin{array}{l} u(x_1,l_2,\ldots,x_n,t)=\varphi_{31}(x_3,\ldots,x_n,t),\ t\in\overline{\Omega},\ 0\le x_i\le l_i, i=\overline{1,n},\ i\ne 2,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u(x_1,\ldots,x_{n-1},l_n,t)=\varphi_{3n}(x_1,\ldots,x_{n-1},t),\ t\in\overline{\Omega},\ 0\le x_i\le l_i, i=\overline{1,n-1}, i\ne n, \end{array}\right\}$$

$$\beta\frac{\partial u(l_1,x_2,\ldots,t)}{\partial x_1}+\alpha u(l_1,x_2,\ldots,t)=\sum_{i=0}^{P}\alpha_i u(\xi_i,x_2,\ldots,t)+\varphi_{31}(x_2,\ldots,t), \tag{2.4}$$

$$t\in\overline{\Omega},\ 0\le x_i\le l_i, i=\overline{2,n},$$

where φ_{2i}, φ_{3i}, $\left(i=\overline{1,n}\right)$ are sufficiently smooth prescribed functions in the corresponding areas, $\{\xi_k\}_{k=0}^{P}$ are given points $0<\xi_0\le\cdots\le\xi_P<l_1$. There is proved the following theorem:

Theorem 2.2. *If in non-local condition (2.4) one of the following two conditions are satisfied: a)* $\alpha_i\ge 0\ \ (i=0,P), \sum_{i=0}^{P}\frac{\alpha_i}{\alpha}\le 1$*; or b)* $\alpha_i\le 0\ \ (i=0,P),\ \sum_{i=0}^{P}\frac{|\alpha_i|}{\alpha}\le 1$ *and there exists the solution of the problem (2.1)-(2.4), then solution is unique.*

3⁰. ITERATION METHOD. For the resolution of the problem stated in paragraph one there is suggested the iteration process,

$$Lu^{k+1}(x,t)=f(x,t),\ (x,t)\in D, \tag{3.1}$$

$$\left.\begin{array}{l} u^{k+1}(x,0,t_2,\ldots,t_m)=\varphi_{11}(x,t_2,\ldots,t_m),\ x\in\overline{G},\ 0\le t_i\le T_i, i=\overline{2,m},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u^{k+1}(x,t_1,\ldots,t_{m-1},0)=\varphi_{1m}(x,t_1,\ldots,t_{m-1}),\ x\in\overline{G},\ 0\le t_i\le T_i, i=\overline{1,m-1}, \end{array}\right\} \tag{3.2}$$

$$u^{k+1}(x,t)=\varphi_2(x,t),\ \ x\in\Gamma^+,\ \ 0\le t_i\le T_i,\ \ i=\overline{1,m}, \tag{3.3}$$

$$\beta\frac{\partial u^{k+1}(x,t)}{\partial\nu}+\alpha u^{k+1}(x,t)=\sum_{i=0}^{P}\alpha_i u^k(x_{\Gamma_i},t)+\varphi_{31}(x,t), \tag{3.4}$$

$$x\in\Gamma^-,\ \ 0\le t_i\le T_i,\ \ i=\overline{1,m},\ \ x_{\Gamma_i}\in\Gamma_i,\ \ i=\overline{0,P},$$

$$k=0,1,2,\ldots$$

and the following theorem is proved:

Theorema 3.1. *If in (2.3.4) condition* $\beta=0$, $\left|\sum_{i=0}^{P}\alpha_i\right|<|\alpha|$ *and there exists the unique solution of the problem (1.1)-(1.4), then the iteration process (3.1)-(3.4) converges to the exact solution of the problem (1.1)-(1.4) with the speed of geometrical progression.*

4⁰. DIFFERENCE SCHEME. There is considered the following problem: there has to be found the function $u(x,t)\in C^{2,1,1}((0,l_1)\times(0,T_1)\times(0,T_2))\cap C^{1,0,0}([0,l_1]\times[0,T_1]\times[0,T_2])$, satisfying the equation

$$\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\partial u(x,t)}{\partial t_1}-\frac{\partial u(x,t)}{\partial t_2}=f(x,t), \tag{4.1}$$

$$x\in(0,l_1),\ t_1\in(0,T_1),\ t_2\in(0,T_2),$$

initial conditions

$$\left.\begin{array}{l} u(x,0,t_2)=\varphi_{11}(x,t_2),\ 0\le x\le l_1,\ 0\le t_2\le T_2,\\ u(x,t_1,0,)=\varphi_{12}(x,t_1),\ 0\le x\le l_1,\ 0\le t_1\le T_1, \end{array}\right\} \tag{4.2}$$

and classical boundary limitations

$$\left.\begin{array}{l} u(0,t_1,t_2)=\varphi_2(t_1,t_2),\ 0\le t_1\le T_1,\ 0\le t_2\le T_2,\\ u(l_1,t_1,t_2)=\varphi_3(t_1,t_2),\ 0\le t_1\le T_1,\ 0\le t_2\le T_2. \end{array}\right\} \tag{4.3}$$

On the regular grid $\omega_{h\tau_1\tau_2}$ there is constructed the following finite-difference problem corresponding to the differential problem (4.1)-(4.3): there has to be found $y_i^{j,k} = y(x_i, t_j, t_k)$ grid function, satisfying the next difference equation,

$$\theta_1 \frac{y_i^{j+1,k+1} - y_i^{j+1,k}}{\tau_2} + (1-\theta_1)\frac{y_i^{j,k+1} - y_i^{j,k}}{\tau_2} + \theta_2 \frac{y_i^{j+1,k+1} - y_i^{j,k+1}}{\tau_1} + (1-\theta_2)\frac{y_i^{j+1,k} - y_i^{j,k}}{\tau_1} =$$
$$= \theta_3\theta_4 L_h y_i^{j+1,k+1} + \theta_3(1-\theta_4)L_h y_i^{j+1,k} + (1-\theta_3)\theta_5 L_h y_i^{j,k+1} + (1-\theta_3)(1-\theta_5)L_h y_i^{j,k} + \tag{4.4}$$
$$+ F_i^{j,k},\ i = \overline{1, N-1},\ j = \overline{0, N_1 - 1},\ k = \overline{0, N_2 - 1},$$

and the following conditions

$$\left.\begin{aligned} y_i^{0,k} &= \varphi_{11i}^{\ 0,k},\ i = \overline{0, N},\ k = \overline{0, N_2},\\ y_i^{j,0} &= \varphi_{12i}^{\ j,0},\ i = \overline{0, N},\ j = \overline{0, N_1}, \end{aligned}\right\} \tag{4.5}$$

$$\left.\begin{aligned} y_0^{j,k} &= \varphi_2^{\ j,k},\ j = \overline{0, N_1},\ k = \overline{0, N_2},\\ y_N^{j,k} &= \varphi_3^{\ j,k},\ j = \overline{0, N_1},\ k = \overline{0, N_2}, \end{aligned}\right\} \tag{4.6}$$

where $y_i^{j,k}$ function is the grid function defined on the $\omega_{h\tau_1\tau_2}$ discrete area corresponding to D, which corresponds to the $u(x, t_1, t_2)$ function, $0 \le \theta_i \le 1$ $(i = \overline{1,5})$ are given parameters, $F_i^{j,k}$, $\varphi_{11i}^{\ 0,k}$, $\varphi_2^{j,k}$ and $\varphi_3^{j,k}$ are respectively grid functions of $[-f(x,t)]$ and those used in the left side of initial and initial-boundary conditions (4.2), (4.3). h, τ_1, τ_2 are steps of regular grid $\omega_{h\tau_1\tau_2}$ correspondingly for x, t_1 and t_2 arguments;

$$L_h y_i^{j,k} = \frac{y_{i+1}^{j,k} - 2y_i^{j,k} + y_{i-1}^{j,k}}{h^2}.$$

The following theorem is true for the scheme (4.4)-(4.6):

Theorem 4.1. *If the function $u(x,t)$ is sufficiently smooth, then the scheme (4.4)-(4.6) approximates the problem (4.1)-(4.3) with the precision of $O(\tau_1 + \tau_2 + h^2)$ order, if in the difference equation (4.4),*

a) $\theta_1 = \theta_2 = \theta_3 = \frac{1}{2}$, $\theta_4 + \theta_5 = 1$, $F_i^{j,k} = -f_i^{j+\frac{1}{2},k+\frac{1}{2}} - \frac{h^2}{12}\frac{\partial^2 f}{\partial x^2} + O(\tau_1^2 + \tau_2^2 + h^2)$, then $\psi_i^{j,k} = O(\tau_1^2 + \tau_2^2 + h^2)$;

b) if $\theta_1 = \theta_2 = \frac{1}{2}$, $\theta_3 = \frac{1}{2} - \frac{h^2}{12\tau_1}$, $\theta_3\theta_4 - \theta_3\theta_5 + \theta_5 = \frac{1}{2} - \frac{h^2}{12\tau_2}$,

$F_i^{j,k} = -f_i^{j+\frac{1}{2},k+\frac{1}{2}} - \frac{h^2}{12}\frac{\partial^2 f}{\partial x^2} + O(\tau_1^2 + \tau_2^2 + h^4)$, then $\psi_i^{j,k} = O(\tau_1^2 + \tau_2^2 + h^4)$,

where $\psi_i^{j,k}$ is an approximation error.

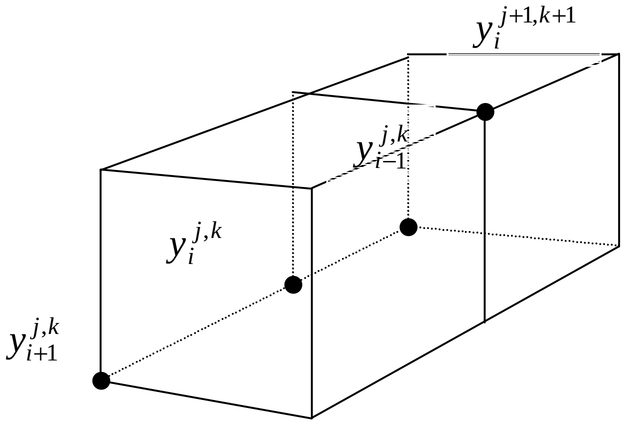
5^0. EXPLICIT AND IMPLICIT FINITE DIFFERENCE SCHEMES. There are considered two explicit and two implicit schemes. They are obtained by selecting the concrete parameters θ_i $(i = \overline{1,5})$ in the problem (4.4)-(4.6).

If $\theta_1 = \theta_2 = \frac{1}{2}$, $\theta_3 = \theta_5 = 0$, then difference equation (2.4.4) takes the form:

$$\frac{1}{2}\frac{y_i^{j+1,k+1} - y_i^{j+1,k}}{\tau_2} + \frac{1}{2}\frac{y_i^{j,k+1} - y_i^{j,k}}{\tau_2} + \frac{1}{2}\frac{y_i^{j+1,k+1} - y_i^{j,k+1}}{\tau_1} + \frac{1}{2}\frac{y_i^{j+1,k} - y_i^{j,k}}{\tau_1} = \tag{5.1}$$
$$= L_h y_i^{j,k} + F_i^{j,k},\ i = \overline{1, N-1},\ j = \overline{0, N_1 - 1},\ k = \overline{0, N_2 - 1},$$

where is assumed, that $\tau_1 = \tau_2 \equiv \tau$. Simulation of difference equation (5.1) consists of five grid points (see pic. 1). The following theorem is true:

Theorem 5.1. *If* $\tau \le \frac{h^2}{2}$*, the finite-difference scheme (5.1), (4.5), (4.6) is stable and its solution converges to the solution of the problem (4.1)-(4.3) in the sense of uniform norm.*



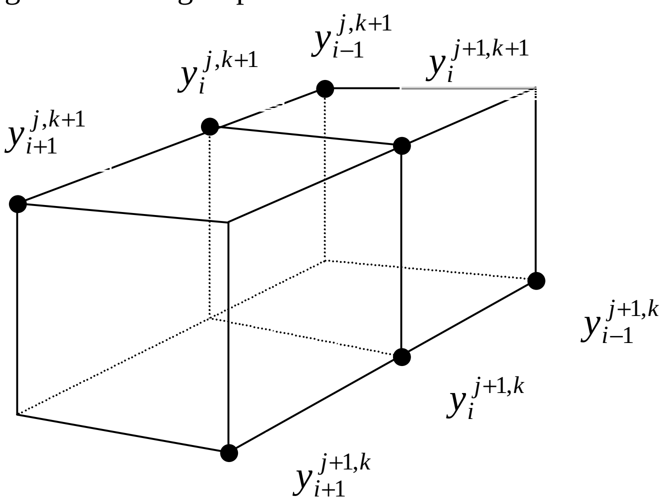
pic. 1.

When $\theta_1 = \theta_2 = 1$, $0 < \theta_3 < 1$, $\theta_4 = 0$, $\theta_5 = 1$, there is obtained the following scheme:

$$\frac{y_i^{j+1,k+1} - y_i^{j+1,k}}{\tau_2} + \frac{y_i^{j+1,k+1} - y_i^{j,k+1}}{\tau_1} = \theta_3 L_h y_i^{j+1,k} + (1-\theta_3)_h y_i^{j,k+1} + F_i^{j,k}, \tag{5.2}$$

$$i = \overline{1, N-1},\ j = \overline{0, N_1 - 1},\ k = \overline{0, N_2 - 1},$$

with simulation consisting from seven grid points.



pic. 2

Stability and convergence issues are covered bye the next theorem:

Theorem 5.2. *If* $\tau_1 \le \frac{h^2}{2\theta_3}$, $\tau_2 \le \frac{h^2}{2(1-\theta_3)}$*, the scheme (5.2), (4.5), (4.6) is stable and its solution converges to the exact solution of the problem (4.1)-(4.3) with the sense of energetic norm.*

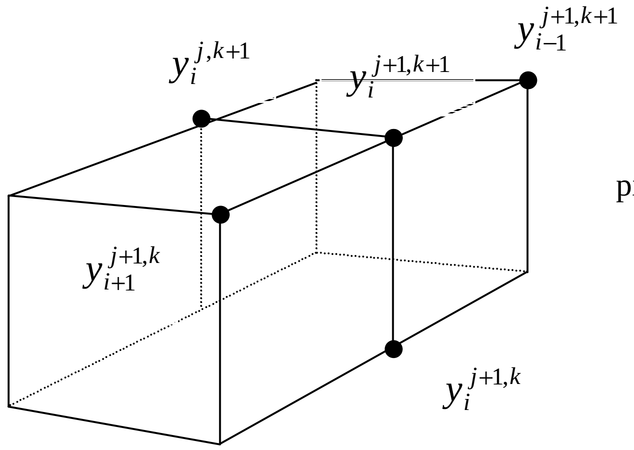
Selecting the parameters in the following way: $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$, there is obtained the equation,

$$\frac{y_i^{j+1,k+1} - y_i^{j+1,k}}{\tau_2} + \frac{y_i^{j+1,k+1} - y_i^{j,k+1}}{\tau_1} = L_h y_i^{j+1,k+1} + F_i^{j,k}, \tag{5.3}$$

$$i = \overline{1, N-1},\ j = \overline{0, N_1 - 1},\ k = \overline{0, N_2 - 1},$$

with simulation which consists of five grid points (see pic. 3).

Theorem 5.3. *The scheme (5.3), (4.5), (4.6) is absolutely stable and the solution converges to the exact solution of the problem (4.1)-(4.3) in the sense of energetic norm.*



pic. 3

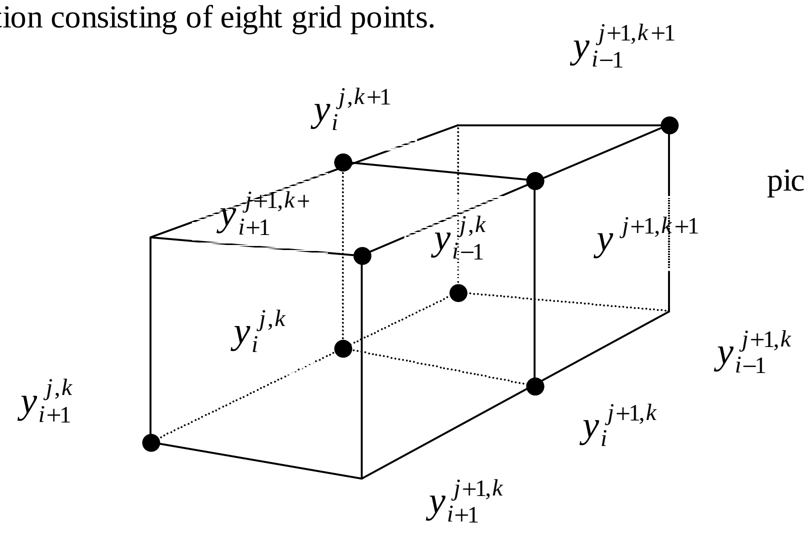
When $\theta_1 = \theta_2 = \theta_3 = 1/2$, $\theta_4 = 1$ and $\theta_5 = 0$, there is obtained another implicit scheme:

$$\frac{1}{2}\frac{y_i^{j+1,k+1} - y_i^{j+1,k}}{\tau_2} + \frac{1}{2}\frac{y_i^{j,k+1} - y_i^{j,k}}{\tau_2} + \frac{1}{2}\frac{y_i^{j+1,k+1} - y_i^{j,k+1}}{\tau_1} + \frac{1}{2}\frac{y_i^{j+1,k} - y_i^{j,k}}{\tau_1} = \tag{5.4}$$

$$= \frac{1}{2} L_h y_i^{j+1,k+1} + \frac{1}{2} L_h y_i^{j,k} + F_i^{j,k},$$

$$\tau_1 = \tau_2 \equiv \tau,\ i = \overline{1, N-1},\ j = \overline{0, N_1 - 1},\ k = \overline{0, N_2 - 1},$$

with simulation consisting of eight grid points.



pic. 4

Theorem 2.5.4. *The scheme (5.4), (4.5), (4.6) is absolutely stable and its solution converges to the exact solution of the problem (4.1)-(4.3) in the sense of energetic norm.*

6⁰. DECOMPOSITION METHOD. There is considered the problem (1.1)-(1.4), when $L \equiv \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, and $G = (0, l_1) \times \cdots \times (0, l_n)$. There is constructed the following decomposition algorithm of parallel count:

$$\left.\begin{array}{l}\sigma_1\left(\sum_{i=1}^{m} y_{1t_i}^{k_1+1,\dots k_i,\dots,k_m+1}\right)=\frac{\partial^2 y_1^{k_1+1,\dots,k_m+1}}{\partial x_1^2}-f_1^{k_1+1,k_2+1,\dots,k_m+1}(x,t),\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \sigma_n\left(\sum_{i=1}^{m} y_{nt_i}^{k_1+1,\dots,k_i,\dots,k_m+1}\right)=\frac{\partial^2 y_n^{k_1+1,\dots,k_m+1}}{\partial x_n^2}-f_n^{k_1+1,k_2+1,\dots,k_m+1}(x,t),\end{array}\right\} \quad (6.1)$$

$$\sum_{i=1}^{n}\sigma_i=1,\sigma_i\geq 0\ t_{k_i}\in\omega_{\tau_i},\left(k_i=\overline{0,N_i-1}\right), j=\overline{1,m},$$

$$\left.\begin{array}{l}y_1^{k_1,k_2+1,\dots,k_m+1}=y_2^{k_1,k_2+1,\dots,k_m+1}=\dots=y_n^{k_1,k_2+1,\dots,k_m+1}=v^{k_1k_2+1,\dots,k_m+1},k_1=\overline{1,N_1},\\ v^{k_1,k_2+1,\dots,k_m+1}=\sum_{i=1}^{n}\sigma_i y_i^{k_1,k_2+1,\dots,k_m+1}, v^{0,k_2+1,\dots,k_m+1}=\varphi_{11}{}^{0,k_2+1,\dots,k_m+1},\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ y_1^{k_1+1,k_2+1,\dots,k_m}=y_2^{k_1+1,k_2+1,\dots,k_m}=\dots=y_n^{k_1+1,k_2+1,\dots,k_m}=v^{k_1+1,k_2+1,\dots,k_m},k_m=\overline{1,N_m},\\ v^{k_1+1,k_2+1,\dots,k_m}=\sum_{i=1}^{n}\sigma_i y_i^{k_1+1,k_2+1,\dots,k_m}, v^{k_1+1,k_2+1,\dots,0}=\varphi_{1m}{}^{k_1+1,k_2+1,\dots,0}\end{array}\right\} \quad (6.2)$$

$$\left.\begin{array}{l}y_1^{k_1+1,k_2+1,\dots,k_m+1}(0,x_2,\dots,t_{k+1})=\varphi_{21}(x_2,\dots,t_{k+1}),0\leq x_i\leq l_i,\left(i=\overline{2,n}\right),\\ \frac{\partial y_1^{k_1+1,k_2+1,\dots,k_m+1}}{\partial x_1}(l_1,x_2,\dots,t_{k+1})+\alpha\, y_1^{k_1+1,k_2+1,\dots,k_m+1}(l_1,x_2,\dots,t_{k+1})=\\ =\sum_{i=0}^{m}\alpha_i y_1^{k_1+1,k_2+1,\dots,k_m+1}(\xi_i,x_2,\dots,t_{k+1})+\varphi_{31}(x_2,\dots,t_{k+1}),\\ 0\leq x_i\leq l_i,\left(i=\overline{1,n},i\neq 1\right), k_i=\overline{0,N_i-1},\end{array}\right\} \quad (6.3)$$

$$\left.\begin{array}{l}y_2^{k_1+1,k_2+1,\dots,k_m+1}(x_1,0,x_3,\dots,t_{k+1})=\varphi_{22}{}^{k_1+1,k_2+1,\dots,k_m+1}(x_1,0,x_3,\dots,t_{k+1}),\\ y_2^{k_1+1,k_2+1,\dots,k_m+1}(x_1,l_2,x_3,\dots,t_{k+1})=\varphi_{32}{}^{k_1+1,k_2+1,\dots,k_m+1}(x_1,l_2,x_3,\dots,t_{k+1})\\ 0\leq x_i\leq l_i,\left(i=\overline{1,n},i\neq 2\right), k_i=\overline{0,N_i-1},\end{array}\right\}$$

$$. \ . \quad (6.4)$$

$$\left.\begin{array}{l}y_n^{k_1+1,k_2+1,\dots,k_m+1}(x_1,\dots,x_{n-1},0,t_{k+1})=\varphi_{2n}{}^{k_1+1,k_2+1,\dots,k_m+1}(x_1,\dots,x_{n-1},0,t_{k+1}),\\ y_n^{k_1+1,k_2+1,\dots,k_m+1}(x_1,l_2,x_3,\dots,t_{k+1})=\varphi_{3n}{}^{k_1+1,k_2+1,\dots,k_m+1}(x_1,\dots,x_{n-1},0,t_{k+1})\\ 0\leq x_i\leq l_i,\left(i=\overline{1,n},i\neq n\right), k=\overline{0,N-1},\end{array}\right\}$$

where $\omega_{\tau_i}=\left\{t_{k_i}:t_{k_i}=k_i\tau_i,k_i=\overline{0,N_i},N_i\tau_i=T_i\right\}$, $t_{k+1}=\left(t_{k_1+1},\dots,t_{k_m+1}\right)$, $y_i^{k_1+1,k_2+1,\dots,k_m+1}=y_i(x,t_{k+1})$, $i=\overline{1,n}$, $k=\overline{0,N}$, $f_i^{k_1+1,k_2+1,\dots,k_m+1}=f_i(x,t_{k+1})$, $i=\overline{1,n}$, $k_i=\overline{0,N_i}$; $y_{jt_i}^{k_1+1,\dots,k_i,\dots,k_m+1}=\frac{y_i^{k_1+1,\dots,k_i+1,\dots,k_m+1}-v^{k_1+1,\dots,k_i,\dots,k_m+1}}{\tau_i}$, $\left(j=\overline{1,n}\right)$; φ_{1i}, $\left(i=\overline{1,m}\right)$, φ_{2i}, $\left(i=\overline{1,n}\right)$ and φ_{3i}, $\left(i=\overline{1,n}\right)$ are given functions. The following theorem takes place:

Theorem 6.1. *If in non-local condition (1.4) one of the two following conditions are satisfied: a)* $\alpha_i\geq 0$ $\left(i=\overline{0,m}\right)$, $\sum_{i=0}^{m}\frac{\alpha_i}{\alpha}\leq 1$ *or b)* $\alpha_i\leq 0$, $\sum_{i=0}^{m}\frac{|\alpha_i|}{\alpha}\leq 1$ $\left(i=\overline{0,m}\right)$, *then (6.7)-(6.10) algorithm of parallel count is stable with respect to initial conditions and right hand function and converges to the exact solution of the initial problem with speed of* $O\left(\tau^{1/2}\right)$.