INTRODUCTION TO GENERALIZED CLASSICAL AND QUANTUM SIGNAL AND SYSTEM THEORIES ON GROUPS AND HYPERGROUPS

Valeriy Labunets
Urals State Tecnical University
Ekaterinburg, Russia
lab@rtf.ustu.ru

"Look," they say, "here is something new!" But no, it has all happened before, long before we have were born.

—Good News Bible, Eccl.1:10

Abstract In this paper we develop two topics in parallel and show their inter- and crossrelation. The first centers on general notions of the classical signal/system theory on finite Abelian hypergroups. The second concerns the quantum hyperharmonic analysis of quantum signals (Hermitean operators associated with classical signals). We study classical and quantum generalized convolution hypergroup algebras of classical and quantum signals.

Keywords: classical and quantum signals/systems, classical and quantum Fourier transforms, Clifford algebra, hypergroups.

Introduction

The main F.Klein idea of the "Erlangen Program" lies in the correspondence of some group to a certain geometry. Thus, a group is the first (basic) notion of geometry and group can be interpreted as some group of symmetries for the geometry. So, in general, every group of transformations (symmetries) determines its own geometry under the F.Klein correspondence **GEO** $= f($**GROUP**$)$. Quantum signal theory is a term referring to a collection of ideas and partial results, loosely held together, which assumes that there are deep connections between the worlds of quantum physics and classical signal/system theory, and that one should try to discover and develop these connections. The general topic of this paper is the following idea. If some algebraic structures arise together in quantum theory and classical signal/system theory in the same context, then one should try to make sense of this for more generalized algebraic structures. Here, the point is not to try to develop alternative theories as substitute models for quantum physics and signal/system theory, but rather to develop a "β-version" of a *unified scheme of general classical and quantum signal/system theory* based on the F.Klein "Erlangen Program". It is known that general building elements of the *Classical and Quantum Signal/System Theories* (**Cl-SST** and **Qu-SST**) are the following: 1) the Abelian group of real numbers **AR**, 2) the classical Fourier transform $\mathcal{F}$, and 3) the complex field **C**, i.e., these theories are associated with the triple $\langle\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\rangle\rangle$. Following F.Klein, we can write

$$\mathbf{Cl\text{–}SST} = f_{cl}\Big(\langle\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\rangle\rangle\Big), \quad \mathbf{Qu\text{–}SST} = f_{qu}\Big(\langle\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\rangle\rangle\Big)$$

for any F.Klein correspondences f_{cl} and f_{qu}, respectively. These correspondences mean that every triple $\langle\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\rangle\rangle$ determines certain theories **Cl–SST** and **Qu–SST**. In this paper we develop a new unified approach to the *Generalized Classical and Quantum Signal/System Theories* (**GCl-SST** and **GQu-SST**). They are based not on the triple $\langle\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\rangle\rangle$, but rather on other Abelian groups and hypergroups, on a large class of orthogonal and unitary transforms (instead of the classical Fourier transform), and involve other fields, rings and algebras (triplet color algebra, multiplet multicolor algebra, hypercomplex commutative algebras, Clifford algebras). In our approach, Generalized Classical and Quantum Signal/System Theories are two functions (correspondences) of a new triple:

$$\mathbf{GCl\text{–}SST} = f_{cl}\Big(\langle\langle \mathbf{HG}, \mathcal{F}, \mathcal{A}\rangle\rangle\Big), \quad \mathbf{GQu\text{–}SST} = f_{qu}\Big(\langle\langle \mathbf{HG}, \mathcal{F}, \mathcal{A}\rangle\rangle\Big),$$

where **HG** is a hypergroup, $\mathcal{F}$ is a unitary transform, and $\mathcal{A}$ is an algebra. When the triple $\langle\langle \mathbf{HG}, \mathcal{F}, \mathcal{A}\rangle\rangle$ is changed the theories **GCl–SST** and **GQu–SST** are changed too. For example, if $\mathcal{F}$ is the classical Fourier transform, **HG** is the group of real numbers **R** and $\mathcal{A}$ is the complex field **C**, then $\langle\langle \mathbf{R}, \mathcal{F}, \mathbf{C}\rangle\rangle$ describes free quantum particles. If $\mathcal{F}$ is the classical

Walsh transform (CWT), **HG** is an abelian dyadic group $\mathbf{Z}_2^n$ and $\mathcal{A}$ is the complex field **C**, then $\langle\langle \mathbf{Z}_2^n, \mathbf{CWT}, \mathrm{A}\rangle\rangle$ describes n-digital quantum registers. If $\mathcal{F}$ is the classical Vilenkin transform (CVT), **HG** is an abelian m-adic group $\mathbf{Z}_m^n$ and $\mathcal{A}$ is the complex field **C**, then $\langle\langle \mathbf{Z}_m^n, \mathbf{CWT}, \mathbf{C}\rangle\rangle$ describes n-digital quantum m-adic registers and so on. Every triple generates a wide class of classical and quantum signal processing methods. We develop these two topics in parallel and show their inter- and crossrelation. We study classical and quantum generalized convolution hypergroup algebras of signals and Hermitian operators. One of the main purposes of this paper is to demonstrate parallelism between the generalized classical hyperharmonic analysis and the generalized quantum hyperharmonic analysis.

1. Generalized classical signal/system theory on hypergroups

1.1 Generalized shift operators

The integral transforms and the signal representation associated with them are important concepts in applied mathematics and in signal theory. The Fourier transform is certainly the best known of the integral transforms and, with the Laplace transform, is also the most useful. Since its introduction by Fourier in the early 1800s, it has found use in innumerable applications. However, the Fourier transform is just one of many ways of signal representation, there are many other transforms of interest. An important aspect of many of these representations is the possibility to extract relevant information from a signal: the information that is actually present but hidden in its complex representation. But these transformations are not efficient analysis tools compared to the ordinary Fourier representation, since the latter is based on such useful and powerful tools of signal theory as linear and nonlinear convolutions, classical and higher-order correlations, invariance with respect to shift, ambiguity and Wigner distributions, etc. The other integral representations have no such tools. The ordinary group shift operators $(T_t^\tau x)(t) := x(t \oplus \tau)$ play the leading role in all the properties and tools of the Fourier transform mentioned above. In order to develop for each orthogonal transform a similar wide set of tools and properties as the Fourier transform has, we associate a family of generalized commutative shift operators with each orthogonal transform. Such families form *commutative hypergroups*. Only in particular cases are these hypergroups well-known Abelian groups. In 1934 F. Marty [1, 2] and H.S. Wall [3, 4] independently introduced the notion of hypergroup.

Let $f(x) : \Omega \longrightarrow \mathcal{A}$ be an $\mathcal{A}$-valued signal, where $\mathcal{A}$ is an algebra. Usually, $\Omega = \mathbf{R}^n \times \mathbf{T}$, or $\Omega = \mathbf{Z}^n \times \mathbf{T}$, where $\mathbf{R}^n$, $\mathbf{Z}^n$ and $\mathbf{Z}_N^n$ are nD vector spaces over $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{Z}_N$, respectively, $\mathbf{T}$ is a compact (temporal) subset of $\mathbf{R}$, $\mathbf{Z}$, or $\mathbf{Z}_N$. Here, $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{Z}_N$ are the real field, the ring of integers, and the ring of integers modulo N, respectively. Let Ω^* be the space dual to Ω. The first one will be called the *spectral domain*, the second one is called the *signal domain* keeping the original notion of $x \in \Omega$ as "time" and $\omega \in \Omega^*$ as "frequency". Let

$$\mathbf{Sig}_0 = \mathbb{L}(\Omega, \mathcal{A}) := \{f(x)|\ f(x) : \Omega \longrightarrow \mathcal{A}\},$$

$$\mathbf{Sp}_0 := \mathbb{L}(\Omega^*, \mathcal{A}) := \{F(\omega)|F(\omega) : \Omega^* \longrightarrow \mathcal{A}\}$$

be two vector spaces of $\mathcal{A}$-valued functions. In the following we assume that the functions satisfy certain general properties so that pathological cases where formulas would not hold are avoided. Let $\{\varphi_\omega(x)\}_{\omega\in\omega^*}$ be an orthonormal system of functions of $\mathbf{Sig}_0$. Then for any function $f(x) \in \mathbf{Sig}_0$ there exists a function $F(\omega) \in \mathbf{Sp}_0$ for which the following equations hold:

$$F(\omega) = \mathcal{CF}\{f\}(\omega) := \int_{x\in\Omega} f(x)\bar{\varphi}_\omega(x)d\mu(x), \qquad (1)$$

$$f(x) = \mathcal{CF}^{-1}\{F\}(x) := \int_{\omega\in\Omega^*} F(\omega)\varphi_\omega(t)d\mu(\omega), \qquad (2)$$

where $\mu(x), \mu(\omega)$ are certain suitable measures on the signal and spectral domains, respectively. The function $F(\omega)$ is called the $\mathcal{CF}$-spectrum of a signal $f(x)$ and expressions (1) and (2) are called the pair of *generalized classical Fourier transforms* (or $\mathcal{CF}$-transforms). In the following we will use the notation $f(x) \underset{\mathcal{CF}}{\longleftrightarrow} F(\omega)$ in order to indicate $\mathcal{CF}$-transform pairs. Along with the "time" and "frequency" domains we will work with "time-time" $\Omega \times \Omega$, "time-frequency" $\Omega \times \Omega^*$, "frequency-time" $\Omega^* \times \Omega$, and "frequency-frequency" $\Omega^* \times \Omega^*$ domains, and with four distributions, which are denoted by double letters $\mathrm{ff}(x, v) \in \mathbf{L}_2(\Omega \times \Omega, \mathcal{A})$, $\mathrm{Ff}(\omega, v) \in \mathbf{L}_2(\Omega^* \times \Omega, \mathcal{A})$, $\mathrm{fF}(x, \nu) \in \mathbf{L}_2(\Omega \times \Omega^*, \mathcal{A})$, and $\mathrm{FF}(\omega, \nu) \in \mathbf{L}_2(\Omega^* \times \Omega, \mathcal{A})$.

The classical shift operators in the "time" and "frequency" domains are defined as $(\widehat{T}_x^v f)(x) := f(x + v)$, $(\widehat{D}_\omega^\nu F)(\omega) := F(\omega + \nu)$. For $f(x) = e^{j\omega x}$ and $F(\omega) = e^{-j\omega x}$, we have $\widehat{T}_x^v e^{j\omega x} = e^{j\omega(x+v)} = e^{j\omega v}e^{j\omega x}$, and $\widehat{D}_\omega^\nu e^{-j\omega x} = e^{-j(\omega+\nu)x} = e^{-j\nu x}e^{-j\omega x}$, i.e., functions $e^{j\omega x}$, $e^{-j\omega x}$ are eigenfunctions of "time"-shift and "frequency"-shift operators $\widehat{T}_x^v$ and $\widehat{D}_\omega^\nu$ corresponding to eigenvalues $\lambda_v = e^{j\omega v}$ and $\lambda_\nu = e^{-j\nu x}$ respectively. We now generalize this result.

Definition 1 *The operators*

$$(\widehat{T}_x^v \varphi_\omega)(x) = \varphi_\omega(x)\varphi_\omega(v), \quad (\widehat{T}_x^{\bar{v}} \varphi_\omega)(x) = \varphi_\omega(x)\bar{\varphi}_\omega(v), \tag{3}$$

$$(\widehat{D}_\omega^\nu \bar{\varphi}_\omega)(x) = \bar{\varphi}_\omega(x)\bar{\varphi}_\nu(x), \quad (\widehat{D}_\omega^{\bar{\nu}} \bar{\varphi}_\omega)(x) = \bar{\varphi}_\omega(x)\varphi_\nu(x). \tag{4}$$

are called the generalized commutative "time" and "frequency"-shift operators (GSOs) respectively.

It is known [5, 6] that two families of time GSOs $\{\widehat{T}_x^v\}_{v\in\Omega}$ and frequency GSOs $\{\widehat{D}_\omega^\nu\}_{\nu\in\Omega^*}$ form two commutative hypergroups. By definition, functions $\varphi_\omega(x)$ are eigenfunctions of GSOs: $\widehat{T}_x^v\varphi_\omega(x) = \varphi_\omega(v)\varphi_\omega(x)$, $\widehat{D}_\omega^\nu\bar{\varphi}_\omega(x) = \bar{\varphi}_\nu(x)\bar{\varphi}_\omega(x)$. For this reason, we can call them the *hypercharacters of the hypergroup.* The idea of a hypercharacter on a hypergroup encompasses characters of locally compact and finite Abelian groups and multiplication formulas for classical orthogonal polynomials. The theory of GSOs was initiated by Levitan [5, 6] and (in the terminology of hypergroup) by Duncl [7] and Jewett [8]. The class of commutative generalized translation hypergroups includes the class of locally compact and finite Abelian groups and semigroups. The theory for these hypergroups looks much like locally compact and finite Abelian group theory. We will show that many well-known harmonic analysis theorems extend to the commutative hypergroups associated with arbitrary Fourier transforms.

For a signal $f(x) \in \mathbf{Sig}_0$ we define its shifted copy by

$$\widehat{T}_x^v f(x) := f(x \boxplus v) = \widehat{T}_x^v \left(\int\limits_{\omega\in\Omega^*} F(\omega)\varphi_\omega(x)d\mu(\omega) \right) =$$

$$\int\limits_{\omega\in\Omega^*} F(\omega)\widehat{T}_x^v\Big(\varphi_\omega\Big)(x)d\mu(\omega) = \int\limits_{\omega\in\Omega^*} [F(\omega)\varphi_\omega(v)]\varphi_\omega(x)d\mu(\omega).$$

Analogously,

$$\widehat{T}_x^{\bar{v}} f(x) := f(x \boxminus v) = \int\limits_{\omega\in\Omega^*} [F(\omega)\bar{\varphi}_\omega(v)]\varphi_\omega(x)d\mu(\omega),$$

$$\widehat{D}_\omega^\nu F(\omega) := F(\omega \oplus \nu) = \int\limits_{x\in\Omega} [f(x)\bar{\varphi}_\nu(x)]\bar{\varphi}_\omega(x)d\mu(x),$$

$$\widehat{D}_\omega^{\bar{\nu}} F(\omega) := F(\omega \ominus \nu) = \int\limits_{x\in\Omega} [f(x)\varphi_\nu(x)]\bar{\varphi}_\omega(x)d\mu(x).$$

Here symbols $\boxplus, \oplus$ and $\boxminus, \ominus$ are the quasisums and quasidifferences, respectively. Obviously

$$\varphi_\omega(x \boxplus v) = \varphi_\omega(x)\varphi_\omega(v), \quad \varphi_\omega(x \boxminus v) = \varphi_\omega(x)\overline{\varphi}_\omega(v),$$

and

$$\varphi_{\omega\oplus\nu}(x) = \varphi_\omega(x)\varphi_\nu(x), \quad \varphi_{\omega\ominus\nu}(x) = \varphi_\omega(x)\overline{\varphi}_\nu(x).$$

We will need the following modulation operators:

$$(\widehat{M}_x^\nu f)(x) := \varphi_\nu(x)f(x), \quad (\widehat{M}_\omega^v F)\omega := \varphi_\omega(v)F(\omega).$$

$$(\widehat{M}_x^{\bar\nu} f)(x) := \bar\varphi_\nu(x)f(x), \quad (\widehat{M}_\omega^{\bar v} F)\omega := \bar\varphi_\omega(v)F(\omega).$$

From the GSOs definition we have:

THEOREM 1 *Shifts and modulations are connected as follows:*

$$\widehat{T}_x^v f(x) = f(x \boxplus v) \underset{\mathcal{CF}}{\longleftrightarrow} F(\omega)\varphi_\omega(v) = \widehat{M}_\omega^v F(\omega),$$

$$\widehat{T}_x^{\bar v} f(x) = f(x \boxminus v) \underset{\mathcal{CF}}{\longleftrightarrow} F(\omega)\bar\varphi_\omega(v) = \widehat{M}_\omega^{\bar v} F(\omega),$$

$$\widehat{M}_x^\nu f(x) = f(x)\bar\varphi_\nu(x) \underset{\mathcal{CF}}{\longleftrightarrow} F(\omega \oplus \nu) = \widehat{D}_\omega^\nu F(\omega),$$

$$\widehat{M}_x^\nu f(x) = f(x)\varphi_\nu(x) \underset{\mathcal{CF}}{\longleftrightarrow} F(\omega \ominus \nu) = \widehat{D}_\omega^{\bar\nu} F(\omega),$$

i.e.,

$$\mathcal{CF}\{\widehat{T}_x^v\}\mathcal{CF}^{-1} = \widehat{M}_\omega^v, \quad \mathcal{CF}\{\widehat{M}_x^\nu\}\mathcal{CF}^{-1} = \widehat{D}_\omega^{\bar\nu}, \tag{5}$$

$$\mathcal{CF}\{\widehat{T}_x^{\bar v}\}\mathcal{CF}^{-1} = \widehat{M}_\omega^{\bar v}, \quad \mathcal{CF}\{\widehat{M}_x^{\bar\nu}\}\mathcal{CF}^{-1} = \widehat{D}_\omega^{\bar\nu}, \tag{6}$$

$$\mathcal{CF}^{-1}\{\widehat{D}_\omega^\nu\}\mathcal{CF} = \widehat{M}_x^{\bar\nu}, \quad \mathcal{CF}^{-1}\{\widehat{M}_\omega^v\}\mathcal{CF} = \widehat{T}_x^v, \tag{7}$$

$$\mathcal{CF}^{-1}\{\widehat{D}_\omega^{\bar\nu}\}\mathcal{CF} = \widehat{M}_x^\nu, \quad \mathcal{CF}^{-1}\{\widehat{M}_\omega^{\bar v}\}\mathcal{CF}_0 = \widehat{T}_x^{\bar v}. \tag{8}$$

The operators are noncommutative because

$$\widehat{M}_x^\nu \widehat{T}_x^v = \bar\varphi_\nu(v)\widehat{T}_x^v \widehat{M}_x^\nu, \quad \widehat{T}_x^v \widehat{M}_x^\nu = \varphi_\nu(v)\widehat{M}_x^\nu \widehat{T}_x^v,$$

$$\widehat{M}_\omega^v \widehat{D}_\omega^\nu = \bar\varphi_\nu(v)\widehat{D}_\omega^\nu \widehat{M}_\omega^v, \quad \widehat{D}_\omega^\nu \widehat{M}_\omega^v = \varphi_\nu(v)\widehat{M}_\omega^v \widehat{D}_\omega^\nu.$$

1.2 Some popular examples of GSOs

EXAMPLE 1 *In this example we consider GSOs on finite cyclic groups. Let* $\Omega = \mathbb{Z}/N$ *be an Abelian cyclic group. The* ND *vector Hilbert space of classical discrete* $\mathcal{A}$*-valued signals is* $\mathbf{Sig}_0 = \{f(x) | f(x) : \mathbb{Z}/N \longrightarrow \mathcal{A}\}$. *The characters of* $\mathbb{Z}/N$ *are discrete harmonic* $\mathcal{A}$*-valued signals* $\chi_\omega(x) = \varepsilon^{\omega x}$, *where* $\omega \in (\mathbb{Z}/N)^* = \mathbb{Z}/N$, *and* ε *is a primitive* N*th root in an algebra* $\mathcal{A}$. *They form a unitary basis in* $\mathbf{Sig}_0$. *The Fourier transform in* $\mathbf{Sig}_0$ *is the discrete Fourier* $\mathcal{A}$*-valued transform*

$$\begin{aligned} f(x) &= \mathcal{CF}_N^{-1}\{F\} = \sum_{\omega \in \mathbb{Z}/N} F(\omega)\varepsilon^{\omega x} \\ F(\omega) &= \mathcal{CF}_N\{f\} = \sum_{x \in \mathbb{Z}/N} f(x)\varepsilon^{-\omega x}. \end{aligned}$$

All Fourier spectra form the ND *vector Hilbert spectral space* $\mathbf{Sp}_0 = \{F(\omega) \mid F(\omega) : \mathbb{Z}/N \longrightarrow \mathcal{A}\}$. *The "time-frequency" and "frequency-time" domains are* $\Omega \times \Omega^* = \Omega \times \Omega = \mathbb{Z}/N \times \mathbb{Z}/N$, *i.e., the phase space is the 2D discrete torus* $\mathbb{Z}/N \times \mathbb{Z}/N$. *The "time" and "frequency"-shift operators* $\widehat{T}_x^v$, $\widehat{D}_\omega^\nu$ *are defined by* $\widehat{T}_x^v f(x) := f(x \oplus v)$, $\widehat{D}_\omega^\nu F(\omega) := f(\omega \oplus \nu)$, *where*

$$\widehat{T}_x^v := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}^v, \quad \widehat{D}_\omega^\nu := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}^\nu$$

and $\oplus$ *is the symbol representing addition modulo* N. *It is obvious that* $\mathcal{CF}_N\{\widehat{T}_x^v\}\mathcal{CF}_N^{-1} = \widehat{M}_\omega^v, \quad \mathcal{CF}_N^{-1}\{\widehat{D}_\omega^\nu\}\mathcal{CF}_N = \widehat{M}_x^{-\nu}$. *Here, modulation operators* $\widehat{M}_x^\nu$ *and* $\widehat{M}_\omega^v$ *are defined by* $\widehat{M}_x^\nu f(x) := \varepsilon^{\nu x} f(x)$, $\widehat{M}_\omega^v F(\omega) := \varepsilon^{\omega v} F(\omega)$, *where*

$$\widehat{M}_x^\nu = \begin{bmatrix} 1 & & & & \\ & \varepsilon^1 & & & \\ & & \varepsilon^2 & & \\ & & & \ddots & \\ & & & & \varepsilon^{N-1} \end{bmatrix}^\nu, \quad \widehat{M}_\omega^v = \begin{bmatrix} 1 & & & & \\ & \varepsilon^1 & & & \\ & & \varepsilon^2 & & \\ & & & \ddots & \\ & & & & \varepsilon^{N-1} \end{bmatrix}^v.$$

The "time"-shift and "frequency"-shift operators induce the following pair of sets of noncommutative Heisenberg–Weyl operators:

$$\mathbb{HW}_x := \left\{ \widehat{\mathcal{E}}_x^{(\nu,v)} = \widehat{M}_x^\nu \widehat{T}_x^v \mid \nu \in \mathbb{Z}/N, \ v \in \mathbb{Z}/N \right\},$$

$$\mathbb{HW}_\omega := \left\{ \widehat{\mathcal{E}}_\omega^{(v,\nu)} = \widehat{M}_\omega^v \widehat{D}_\omega^\nu \mid v \in \mathbb{Z}/N,\ \nu \in \mathbb{Z}/N \right\}.$$

They act on $\mathbf{Sig}_0$ *and* $\mathbf{Sp}_0$ *by the following rules:*

$$\widehat{\mathcal{E}}_x^{(\nu,v)} f(x) := \widehat{M}_x^\nu \widehat{T}_x^v f(x) = \varepsilon^{\nu x} f(x \oplus v),$$

$$\widehat{\mathcal{E}}_\omega^{(v,\nu)} F(\omega) := \widehat{M}_\omega^v \widehat{D}_\omega^\nu F(\omega) = \varepsilon^{v\omega} F(\omega \oplus \nu).$$

□

EXAMPLE 2 *Let* $\Omega_\mathbf{N}$ *and* $\Omega_\mathbf{N}^*$ *be two versions of a finite Abelian group of order* $\mathbf{N} := N_1 N_2 \cdots N_n$. *The fundamental structure theorem for finite Abelian groups implies that we may write* $\Omega_\mathbf{N}$ *and* $\Omega_\mathbf{N}^*$ *as the direct sums of cyclic groups, i.e.,* $\Omega_\mathbf{N} = \bigoplus_{l=1}^{m} \mathbb{Z}/N_l$, *and* $\Omega_\mathbf{N}^* = \bigoplus_{l=1}^{m} \mathbb{Z}^*/N_l$, *where both* $\mathbb{Z}/N_l$ *and* $\mathbb{Z}^*/N_l$ *are identified with the integers* $0, 1, \ldots, N_l - 1$ *under addition modulo* N_l. *Group elements* $x \in \Omega_\mathbf{N}$ *and* $\omega \in \Omega_\mathbf{N}^*$ *are identified with points* $x = (x_1, x_2, \ldots, x_m)$ *and* $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ *of the mD discrete torus, respectively. Let us embed finite groups* $\Omega_\mathbf{N}$ *and* $\Omega_\mathbf{N}^*$ *into two discrete segments* $\Omega_\mathbf{N} \longrightarrow \Omega := [0, \mathbf{N}-1]$, $\Omega_\mathbf{N}^* \longrightarrow \Omega^* := [0, \mathbf{N}-1]^*$ *using a mixed-radix number system*

$$x = \sum_i x_i \left(\prod_{j=0}^{i-1} N_j \right), \quad \omega = \sum_i \omega_i \left(\prod_{j=0}^{i-1} N_j \right).$$

The weights of x_1 *and* ω_1 *are unity* ($N_0 = 1$.) *The group addition induces "exotic" shifts in the segments* $\Omega := [0, \mathbf{N}-1]$ *and* $\Omega^* := [0, \mathbf{N}-1]^*$, *which we will denote as* $\underset{\mathbf{N}}{\oplus}$. *If* $x = (x_1, \ldots, x_m)$, $v = (v_1, \ldots, v_m)$ *and* $\omega = (\omega_1, \ldots, \omega_m)$, $\nu = (\nu_1, \ldots, \nu_m)$, *then*

$$x \underset{\mathbf{N}}{\oplus} v = (x_1, \ldots, x_m) \underset{\mathbf{N}}{\oplus} (v_1, \ldots, v_m) = (x_1 \underset{N_1}{\oplus} v_1, \ldots, x_m \underset{N_m}{\oplus} v_m)$$

and

$$\omega \underset{\mathbf{N}}{\oplus} \nu = (\omega_1, \ldots, \omega_m) \underset{\mathbf{N}}{\oplus} (\nu_1, \ldots, \nu_m) = (\omega_1 \underset{N_1}{\oplus} \nu_1, \ldots, \omega_m \underset{N_m}{\oplus} \nu_m).$$

The Fourier transforms in the space of all $\mathcal{A}$*-valued signals defined on the finite Abelian group* $\Omega_\mathbf{N} = \mathbb{Z}/N_1 \times \mathbb{Z}/N_2 \times \ldots \times \mathbb{Z}/N_m$ *in the form of* $\Omega = [0, \mathbf{N}-1]$ *have a great interest for digital signal processing. Denote this space by* $\mathbf{Sig}_0 = \mathbb{L}(\Omega, \mathcal{A})$. *Let* ε_{N_l} *be a primitive* $\mathcal{A}$*-valued* N_l*-th root. The set of all characters of the group* $\Omega_\mathbf{N}$ *can be described by* $\chi_\omega(x) = \chi_{\omega_1}(x_1) \cdots \chi_{\omega_m}(x_m) = \varepsilon_1^{\omega_1 x_1} \cdots \varepsilon_m^{\omega_m x_m}$. *They form an orthogonal basis* *in the signal space* $\mathbb{L}(\Omega, \mathcal{A})$. *The Fourier transform of a signal* $f(x) \in \mathbb{L}(\Omega, \mathcal{A})$ *is defined as*

$$F(\omega) = \mathcal{CF}_{\mathbf{N}}\{f\}(\omega) = \sum_{t\in\Omega} f(x)\overline{\chi}_\omega(x), \quad \omega \in \Omega^*. \tag{9}$$

The inverse Fourier transform is

$$f(x) = \mathcal{CF}_{\mathbf{N}}^{-1}\{F\}(x) = \frac{1}{N}\sum_{\omega\in\Omega^*} F(\omega)\chi_\omega(x), \quad t \in \Omega. \tag{10}$$

The set of all functions $F(\omega)$ *forms the spectral space* $\mathbf{Sp}_0 = \mathbb{L}(\Omega^*, \mathcal{A})$. *The "time" and "frequency"-shift operators* $\widehat{T}_x^v$, $\widehat{D}_\omega^\nu$ *are defined by*

$$\widehat{T}_x^v f(x) := f(x \underset{\mathbf{N}}{\oplus} v), \quad \widehat{D}_\omega^\nu F(\omega) := f(\omega \underset{\mathbf{N}}{\oplus} \nu),$$

where

$$\widehat{T}_x^v = \widehat{T}_{(x_1,x_2,\ldots,x_m)}^{(v_1,v_2,\ldots,v_m)} = \widehat{T}_{x_1}^{v_1} \otimes \widehat{T}_{x_2}^{v_2} \otimes \ldots \otimes \widehat{T}_{x_m}^{v_m},$$

and

$$\widehat{D}_\omega^\nu = \widehat{D}_{(\omega_1,\omega_2,\ldots,\omega_m)}^{(\nu_1,\nu_2,\ldots,\nu_m)} = \widehat{D}_{\omega_1}^{\nu_1} \otimes \widehat{D}_{\omega_2}^{\nu_2} \otimes \ldots \otimes \widehat{D}_{\omega_m}^{\nu_m}.$$

Obviously,

$$\begin{aligned} \mathcal{CF}_{\mathbf{N}}\{\widehat{T}_x^v\}\mathcal{CF}_{\mathbf{N}}^{-1} &= \widehat{M}_\omega^v \\ \mathcal{CF}_{\mathbf{N}}^{-1}\{\widehat{D}_\omega^\nu\}\mathcal{CF}_{\mathbf{N}} &= \widehat{M}_x^{-\nu}. \end{aligned}$$

Here, modulation operators $\widehat{M}_x^\nu$ *and* $\widehat{M}_\omega^v$ *are defined by* $\widehat{M}_x^\nu f(x) := \chi_\omega(x)f(x)$ *and* $\widehat{M}_\omega^v F(\omega) := \chi_\omega(x)F(\omega)$, *where*

$$\widehat{M}_x^\nu = \widehat{M}_{(x_1,x_2,\ldots,x_m)}^{(\nu_1,\nu_2,\ldots,\nu_m)} = \widehat{M}_{x_1}^{\nu_1} \otimes \widehat{M}_{x_2}^{\nu_2} \otimes \ldots \otimes \widehat{M}_{x_m}^{\nu_m}$$

and

$$\widehat{M}_\omega^v = \widehat{M}_{(\omega_1,\omega_2,\ldots,\omega_m)}^{(v_1,v_2,\ldots,v_m)} = \widehat{M}_{\omega_1}^{v_1} \otimes \widehat{M}_{\omega_2}^{v_2} \otimes \ldots \otimes \widehat{M}_{\omega_m}^{v_m}.$$

□

EXAMPLE 3 *Let* $\Omega = [a, b]$, $\Omega^* = \{0, 1, 2, ...\} := \mathbf{N}_0$, *and let* $\varphi_k(t) \equiv p_k(t)$ *be a family of classical orthogonal polynomials. Then*

$$F(k) := \int_a^b f(t)p_k(t)\varrho(t)dt, \quad f(t) := \sum_{k=0}^{\infty} h_k^{-1}F(k)p_k(t) \tag{11}$$

is the pair of generalized Fourier transforms, where $k \in \mathbf{N}_0$, $t \in [a, b]$, $\varrho(t)dt = d\mu(t)$ *and* $d\mu(k) = h_k^{-1}$ *are measures on the signal and spectral* *domains respectively. We consider special cases of classical orthogonal polynomials. Case 1. Let* $\Omega = [-1,+1]$, $\varrho(t) = (1-t)^{\alpha}\ (1+\ t)^{\beta}$, $\alpha > \beta - 1$ *and* $Jac_k^{(\alpha,\beta)}(t)$ *be* (α,β)*-Jacobi polynomials. In this case, generalized Fourier transforms for each* α *and* β *are the Fourier–Jacobi transforms*

$$ {}^{(\alpha,\beta)}F(k) = {}^{(\alpha,\beta)}\mathcal{CF}\{f\}(k) = \int_{-1}^{+1} f(t) Jac_k^{(\alpha,\beta)}(t)(1-t)^{\alpha}(1+t)^{\beta}dt, \quad (12) $$

$$ f(t) := {}^{(\alpha,\beta)}\mathcal{CF}^{-1}\{F\}(k) = \sum_{k=0}^{\infty} h_k^{-1}\ {}^{(\alpha,\beta)}F(k) Jac_k^{(\alpha,\beta)}(t) \quad (13) $$

for special constants h_k. *If* $\alpha > \beta > -\frac{1}{2}$ *then the multiplication formula for* (α,β)*-Jacobi polynomials is*

$$ Jac_k^{(\alpha,\beta)}(\tau) Jac_k^{(\alpha,\beta)}(t) = P_k^{(\alpha,\beta)}(t \boxplus \tau) = \int_0^1\int_0^{\pi} Jac_k^{(\alpha,\beta)}\Bigg[\frac{1}{2}(1+\tau)(1+t)+ $$

$$ \frac{1}{2}(1-\tau)(1-t)s^2 + \sqrt{(1-\tau^2)(1-t^2)}s\cos\theta - 1\Bigg] d\mu(s,\theta), \quad (14) $$

where $d\mu(s,\theta) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}(1-s^2)^{\alpha-\beta-1}s^{2\beta+1}(\sin\theta)^{2\beta}dsd\theta$. *There follows*

$$ (T_t^{\tau} f)(t) = f(t \boxplus \tau) = \int_0^1\int_0^{\pi} f\Bigg[\frac{1}{2}(1+\tau)(1+t)+ $$

$$ +\frac{1}{2}(1-\tau)(1-t)s^2 + \sqrt{(1-\tau^2)(1-t^2)}s\cos\theta - 1\Bigg] d\mu(s,\theta). \quad (15) $$

Case 2. If $\alpha = \beta = 0$ *then* $\{Jac_k^{(0,0)}(t)\}_{k=0}^{\infty} = \{Leg_k(t)\}_{k=0}^{\infty}$ *is the Legendre basis. From (15) we obtain the Legendre GSOs*

$$ (T_t^{\tau} f)(t) = f(t \boxplus \tau) = \frac{1}{2\pi}\int_{-1}^{1} f\left(\tau t + \sqrt{(1-\tau^2)(1-t^2)}s\right)(1-s^2)^{-1/2}ds, \quad (16) $$

associted with the Legendre transform. Case 3. If $\alpha = \beta = -0.5$, *then* $\{Jac_k^{(-0.5,-0.5)}(t)\}_{k=0}^{\infty} = \{Ch_k(t)\}_{k=0}^{\infty}$ *is the Legendre basis. In this case,* $\Omega = (-1,1)$, $\varrho(t) = (1-t^2)^{-1/2}$, $h_0 = \frac{\pi}{2}$, $h_n = \pi$, $n \in \mathbf{N}_0 \equiv \Omega^*$. *For the Chebyshev polynomials the following multiplication formula is known:*

$$ Ch_n(\tau) Ch_n(t) = Ch_n(t \boxplus \tau) = $$

$$\frac{1}{2}\left[Ch_n\left(\tau t+\sqrt{(1-\tau^2)(1-t^2)}\right)+Ch_n\left(\tau t-\sqrt{(1-t^2)(1-t^2)}\right)\right]. \tag{17}$$

Therefore,

$$(T_t^\tau f)(t)=f(t\boxplus\tau)=$$
$$\frac{1}{2}\left[f\left(t\tau+\sqrt{(1-\tau^2)(1-t^2)}\right)+f\left(t\tau-\sqrt{(1-\tau^2)(1-t^2)}\right)\right]. \tag{18}$$

□

EXAMPLE 4 *Finally, we consider the infinite interval $\Omega=(-\infty,+\infty)$. Let us introduce the signal and the spectrum spaces*

$$\mathbb{L}_2\Big(\mathbb{R},\mathbb{C},w(t)\Big)=\left\{f(t)\;\Big|\;\Big(f(t):\mathbb{R}\longrightarrow\mathbb{C}\Big)\&\Big(\int\limits_{-\infty}^{+\infty}|f(t)|^2w(t)dt<\infty\Big)\right\},$$

$$\mathbb{L}_2(\mathbb{N},\mathbb{C},\mu_n)=\left\{F(n)\;\Big|\;\Big(F(n):\mathbb{N}\longrightarrow\mathbb{C}\Big)\&\Big(\sum_{n\in\mathbb{N}}w_n|F(n)|^2<\infty\Big)\right\},$$

with the scalar products

$$(f,g):=\int\limits_{-\infty}^{+\infty}f(t)g(t)e^{-t^2/2}\,dt,\quad (F,G)=\sum_{n\in\mathbb{N}}\frac{1}{2^nn!\sqrt{\pi}}F(n)G(n),$$

where $\Omega^*=\mathbb{N}=\{0,1,2,\dots,\}$, $d\mu(t)=w(t)dt$, $w(t)=e^{-t^2/2}$, and $w_n=1/2^nn!\sqrt{\pi}$. In this case, the generalized classical Fourier transform of a signal $f(t)\in\mathbb{L}_2\Big(\mathbb{R},\mathbb{C},e^{-t^2/2}\Big)$ is the Fourier–Hermite transform

$$F(n)=\mathcal{CF}\{f\}(n)=\int_{-\infty}^{+\infty}f(t)Her_n(t)e^{-t^2/2}\,dt,$$

where

$$f(t)=\mathcal{CF}^{-1}\{F\}(n)=\sum_{n=0}^{\infty}\frac{1}{2^nn!\sqrt{\pi}}F(n)Her_n(t),$$

where $Her_n(t)$ are Hermite polynomials. Since

$$Her_k(t)Her_k(\tau)=Her_k(t\boxplus\tau)=\frac{(-1)^k\Gamma(k+(3/2))^{2k+1}}{\sqrt{\pi}}\times$$

$$\times\int_0^\pi Her_k\Big[(t^2+\tau^2+2t\tau\cos\varphi)^{1/2}\exp(-t\tau\sin\varphi)\sin\varphi J_0(t\tau\sin\varphi)d\varphi\Big], \tag{19}$$

then

$$T_t^\tau f(t) = f(t \boxplus \tau) =$$
$$= \int_0^\pi f\left(\sqrt{t^2+\tau^2+2t\tau\cos\varphi}\right) e^{-t\tau\cos\varphi} \sin\varphi J_0(t\tau\sin\varphi) d\varphi, \qquad (20)$$

where $J_0(.)$ *is the Bessel function.* □

1.3 Generalized convolutions and correlations

It is well known that stationary linear dynamic systems (LDS) are described by convolution integrals. Using the GSO notion, we can formally generalize the notions of convolution and correlation [11]–[18].

Definition 2 *The following functions*

$$y(x) := (h\Diamond f)(x) = \int\limits_{v\in\Omega} h(v) f(x \boxminus v) d\mu(v), \qquad (21)$$

$$Y(\omega) := (H\heartsuit F)(\omega) = \int\limits_{\nu\in\Omega^*} H(\nu) F(\omega \ominus \nu) d\mu(\nu) \qquad (22)$$

are called the $\Diamond$ *and* $\heartsuit$*-convolutions respectively.*

The spaces $\mathbf{Sig}_0$ and $\mathbf{Sp}_0$ equipped with multiplications $\Diamond$ and $\heartsuit$ form commutative signal and spectral convolution algebras $\langle\langle \mathbf{Sig}_0, \Diamond\rangle\rangle$ and $\langle\langle \mathbf{Sp}_0, \heartsuit\rangle\rangle$, respectively.

Definition 3 *The expressions*

$$(f\clubsuit g)(v) := \int\limits_{x\in\Omega} f(x)\overline{g}(x \boxminus v) d\mu(x), \qquad (23)$$

$$(F\spadesuit G)(\nu) := \int\limits_{\omega\in\Omega^*} F(\omega)\overline{G}(\omega \ominus \nu) d\mu(\omega) \qquad (24)$$

are referred to as the cross $\clubsuit$ *and* $\spadesuit$*-correlation functions of signals* f, g *and spectra* F, G*, respectively. If* $f = g$ *and* $F = G$*, then the crosscorrelation functions are called the* $\clubsuit$ *and* $\spadesuit$*-autocorrelation functions.*

The measures indicating the similarity between fF-distributions and Ff-distributions and their time and frequency-shifted versions are their crosscorrelation functions.

Definition 4 *The expressions*

$$(fF\clubsuit\spadesuit gG)(v,\nu) := \int\limits_{t\in\Omega}\int\limits_{\omega\in\Omega^*} fF(x,\omega)\overline{gG}(x\boxminus v, \omega\ominus\nu) d\mu(x) d\mu(\omega), \qquad (25)$$

$$(Ff\spadesuit\clubsuit Gg)(\nu,v) := \int\limits_{\nu\in\Omega^*}\int\limits_{v\in\Omega} Ff(\omega,t)\overline{Gg}(\omega\ominus\nu, t\boxminus v)d\mu(x)d\mu(\omega). \quad (26)$$

are referred to as the $\clubsuit\spadesuit$ *and* $\spadesuit\clubsuit$*-crosscorrelation functions of the distributions respectively. If* $fF(x,\omega) = gG(x,\omega)$ *and* $Ff(\omega,t) = Gg(\omega,t)$*, then the crosscorrelation functions are called the autocorrelation functions.*

THEOREM 2 *Generalized classical Fourier transforms (1) and (2) map* $\diamondsuit$ *and* $\heartsuit$*-convolutions and* $\clubsuit$ *and* $\spadesuit$*-correlations into the products of spectra and signals, respectively,*

$$\mathcal{CF}\{h\diamondsuit f\} = \mathcal{CF}\{h\}\,\mathcal{CF}\{f)\}\,, \quad \mathcal{CF}^{-1}\{H\heartsuit F\} = \mathcal{CF}^{-1}\{H\}\,\mathcal{CF}^{-1}\{F\}$$

$$\mathcal{CF}\{f\clubsuit g\} = \mathcal{CF}\{f\}\,\overline{\mathcal{CF}\{g\}}, \quad \mathcal{CF}^{-1}\{F\spadesuit G\} = \mathcal{CF}^{-1}\{F\}\,\overline{\mathcal{CF}^{-1}\{G\}}$$

Taking special forms of the GSOs, one can obtain known types of convolutions and crosscorrelations: arithmetic, cyclic, dyadic, m-adic, etc. Signal and spectral algebras have many of the properties associated with classical group convolution algebras. Many of them are catalogued in [9]–[19].

1.4 Generalized ambiguity functions and Wigner distributions

The Wigner distribution was introduced in 1932 by E. WIGNER [20] in the context of quantum mechanics. There he defined the probability distribution function of simultaneous values of the spatial coordinates and impulses. Wigner's idea was introduced in signal analysis in 1948 by J. VILLE [21], but it did not receive much attention there until 1953 when P. WOODWARD [22] reformulated it in the context of radar theory. Woodward proposed treating the question of radar signal ambiguity as part of the question of target resolution. For that, he introduced a function that described the correlation between a radar signal and its Doppler-shifted and time-translated version:

$$\mathrm{AW}^a[f](\nu,v) = \int\limits_{-\infty}^{+\infty} f(x)\bar{f}(x-v)e^{-j\nu x}dx = \mathop{\mathcal{CF}}_{x\to\nu}\{ff^a(x,v)\},$$

where $ff^a(x,v) := f(x)\bar{f}(x-v)$. The distribution $\mathrm{AW}^a[f](\nu,v)$ is called the *asymmetric Woodward ambiguity function*. It describes the local ambiguity of locating targets in range (time delay v) and in velocity

(Doppler frequency ν). Its absolute value is called the *uncertainty function* since it is related to the *uncertainty principle* of radar signals.

The next time-frequency distribution is the so-called *symmetric Woodward ambiguity function*:

$$\mathrm{AW}^s[f](\nu,v) := \underset{x\to\nu}{\mathcal{CF}}\left\{f\left(x+\frac{v}{2}\right)\bar{f}\left(x-\frac{v}{2}\right)\right\} = \underset{x\to\nu}{\mathcal{CF}}\left\{ff^s(x,v)\right\}, \quad (27)$$

where $ff^s(x,v) := f\left(x+\frac{v}{2}\right)\bar{f}\left(x-\frac{v}{2}\right)$. Analogously, we have expressions for computing $\mathrm{AW}^s[F](v,\nu)$ in the frequency domain

$$\mathrm{AW}^a[F](\nu,v) = \underset{v\leftarrow\omega}{\mathcal{CF}^{-1}}\{F(\omega)\bar{F}(\omega-\nu)\} = \underset{v\leftarrow\omega}{\mathcal{CF}^{-1}}\{FF^a(\nu,\omega)\},$$

$$\mathrm{AW}^s[F](\nu,v) = \underset{v\leftarrow\omega}{\mathcal{CF}^{-1}}\{F\left(\omega+\frac{\nu}{2}\right)\bar{F}\left(\omega-\frac{\nu}{2}\right)\} = \underset{v\leftarrow\omega}{\mathcal{CF}^{-1}}\{FF^s(\nu,\omega)\}.$$

If $F = \mathcal{CF}\{f\}$, then from Parseval's relation we obtain

$$\mathrm{AW}^a[f](\nu,v) = \mathrm{AW}^a[F](\nu,v) \quad \text{and} \quad \mathrm{AW}^s[f](\nu,v) = \mathrm{AW}^s[F](\nu,v).$$

For this reason, we shall denote $\mathrm{AW}^a[f](\nu,v)$, $\mathrm{AW}^a[F](\nu,v)$ by $\mathrm{AW}^a(\nu,v)$ and $\mathrm{AW}^s[f](\nu,v)$, $\mathrm{AW}^s[F](\nu,v)$ by $\mathrm{AW}^s(\nu,v)$. Further, we use the symbol $\mathrm{AW}(\nu,v)$ for both $\mathrm{AW}^a(\nu,v)$ and $\mathrm{AW}^s(\nu,v)$.

Important examples of time-frequency distributions are the so-called *asymmetrical and symmetrical Wigner–Ville distributions*. They can be defined as the 2D symplectic Fourier transform of $\mathrm{AW}^a[f](\nu,v)$ and $\mathrm{AW}^s[f](\nu,v)$, respectively,

$$\mathrm{WV}^a[f](x,\omega) = \underset{x\leftarrow\nu}{\mathcal{CF}^{-1}}\underset{\omega\leftarrow v}{\mathcal{CF}}\{\mathrm{AW}^a[f](\nu,v)\} = f(x)\overline{F}(\omega)e^{-j\omega x}, \quad (28)$$

$$\mathrm{WV}^s[f](x,\omega) = \underset{x\leftarrow\nu}{\mathcal{CF}^{-1}}\underset{\omega\leftarrow v}{\mathcal{CF}}\{\mathrm{AW}^s[f](\nu,v)\} = \int_{-\infty}^{\infty} f\left(x+\frac{v}{2}\right)\bar{f}\left(x-\frac{v}{2}\right)e^{-j\omega v}dv. \quad (29)$$

The 2D symplectic Fourier transform in (29) and (28) can be also viewed as two sequentially performed 1D transforms with respect to v and ν. The transform with respect to ν yields the *temporal autocorrelation functions*

$$\mathrm{ff}^a(x,v) = \underset{x\leftarrow\nu}{\mathcal{CF}^{-1}}\{\mathrm{AW}^s[f](\nu,v)\} = f(x)f(x-v),$$

$$\mathrm{ff}^s(x,v) = \underset{x\leftarrow\nu}{\mathcal{CF}^{-1}}\{\mathrm{AW}^s[f](\nu,v)\} = f\left(x+\frac{v}{2}\right)f\left(x-\frac{v}{2}\right).$$

The transform with respect to ν yields the *frequency autocorrelation functions*

$$\mathrm{FF}^a(\nu,\omega) = \underset{\omega\leftarrow v}{\mathcal{CF}}\{\mathrm{AW}^s[F](\nu,v)\} = F(\omega)F(\omega-\nu),$$

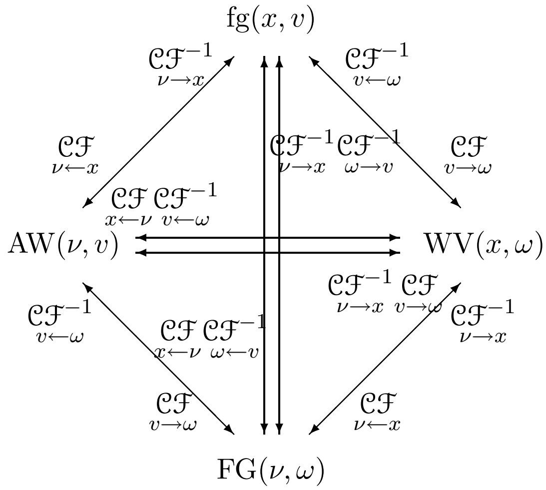

Figure 1. Diagram of relations between the different generalized 2D distributions

$$\mathrm{FF}^s(\nu,\omega) = \underset{\omega\leftarrow v}{\mathcal{CF}}\{\mathrm{AW}^s[F](\nu,v)\} = F\left(\omega+\frac{\nu}{2}\right)F\left(\omega-\frac{\nu}{2}\right).$$

We can formally generalize the notions of cross-ambiguity functions and Wigner-Ville distributions using the GSO notion.

DEFINITION 5 *The symmetric and asymmetric generalized Woodward distributions (cross-ambiguity functions) of two signals f, g and two spectra F, G are defined by*

$$AW^s[f,g](\nu,v) = \underset{\nu\leftarrow x}{\mathcal{CF}}\left\{fg^s\right\} = \int\limits_{x\in\Omega}\left[f\left(x\boxplus\frac{v}{2}\right)\bar{g}\left(x\boxminus\frac{v}{2}\right)\right]\bar{\varphi}_\nu(x)d\mu(x),$$

$$AW^s[F,G](\nu,v) = \underset{v\leftarrow\omega}{\mathcal{CF}^{-1}}\left\{FG^s\right\} = \int\limits_{\omega\in\Omega^*}\left[F\left(\omega\oplus\frac{\nu}{2}\right)\bar{G}\left(\omega\ominus\frac{\nu}{2}\right)\right]\varphi_\omega(v)d\mu(\omega),$$

$$AW^a[f,g](\nu,v) = \underset{\nu\leftarrow x}{\mathcal{CF}}\left\{fg^a\right\} = \int\limits_{x\in\Omega}\left[f(x)\bar{g}(x\boxminus v)\right]\bar{\varphi}_\nu(x)d\mu(x),$$

$$AW^a[F,G](\nu,v) = \underset{v\leftarrow\omega}{\mathcal{CF}^{-1}}\left\{FG^a\right\} = \int\limits_{\omega\in\Omega^*}\left[F(\omega)\bar{G}(\omega\ominus\nu)\right]\varphi_\omega(v)d\mu(\omega).$$

DEFINITION 6 *The generalized symmetric and asymmetric Wigner-Ville distributions of two signals f, g and two spectra F, G are defined by*

$$WV^s[f,g](x,\omega) = \underset{\omega\leftarrow v}{\mathcal{CF}}\left\{fg^s\right\} = \int\limits_{v\in\Omega}\left[f\left(x\boxplus\frac{v}{2}\right)\bar{g}\left(x\boxminus\frac{v}{2}\right)\right]\bar{\varphi}_\omega(v)d\mu(v),$$

$$WV^s[F,G](x,\omega) = \underset{x\leftarrow\nu}{\mathcal{CF}^{-1}}\{FG^s\} = \int\limits_{\nu\in\Omega^*}\left[F\Big(\omega\oplus\frac{\nu}{2}\Big)\bar{G}\left(\omega\oplus\frac{\nu}{2}\Big)\right]\varphi_\nu(x)d\mu(\nu),$$

$$WV^a[f,g](x,\omega) := \underset{\omega\leftarrow v}{\mathcal{CF}}\{fg^a\} = f(x)\overline{F}(\omega)\bar{\varphi}_\omega(x),$$

$$WV^a[F,G](x,\omega) := \underset{x\leftarrow\nu}{\mathcal{CF}^{-1}}\{FG^a\} = F(\omega)\bar{f}(x)\varphi_\omega(x).$$

Figure 1 is a flowchart relating

$$\mathrm{AW}[f,g](\nu,v), \qquad \mathrm{AW}[F,G](\nu,v)$$

and

$$\mathrm{WV}[f,g](x,\omega), \qquad \mathrm{WV}[F,G](x,\omega)$$

and

$$fg(x,v), \qquad FG(\nu,\omega).$$

We can construct two vector Hilbert spaces of "time-frequency" and "frequency-time" distributions

$$\mathbf{WV} := \{\mathrm{WV}(x,\omega) \mid \mathrm{WV}(x,\omega) : \Omega\times\Omega^* \longrightarrow \mathcal{A}\},$$

$$\mathbf{AW} := \{\mathrm{AW}(\nu,v) \mid \mathrm{AW}(\nu,v) : \Omega^*\times\Omega \longrightarrow \mathcal{A}\}.$$

Definition 7 *The generalized Woodward–Gabor ambiguity transforms (or short-time and short-frequency generalized Fourier transforms)* $\mathcal{WG}_g$ *and* $\mathcal{WG}_G$ *associated with functions* g *and* G *are defined as the following mappings:*

$$\mathcal{WG}_g: \ \mathbb{L}(\Omega,\mathcal{A}) \longrightarrow \mathbb{L}(\Omega^*\times\Omega,\mathcal{A}), \quad \mathcal{WG}_G: \ \mathbb{L}(\Omega^*,\mathcal{A}) \longrightarrow \mathbb{L}(\Omega^*\times\Omega,\mathcal{A})$$

given by

$$\mathcal{WG}_g\{f\}(\nu,v) := AW[f,g](\nu,v), \quad \mathcal{WG}_G\{F\}(\nu,v) := AW[F,G](v,\nu).$$

Definition 8 *The generalized Wigner-Ville transforms* $\mathcal{WV}_g$ *and* $\mathcal{WV}_G$ *associated with functions* g *and* G *are defined as mappings*

$$\mathcal{WV}_g \ : \ \mathbb{L}(\Omega,\mathcal{A}) \longrightarrow \mathbb{L}(\Omega\times\Omega^*,\mathcal{A}), \quad \mathcal{WV}_G \ : \ \mathbb{L}(\Omega^*,\mathcal{A}) \longrightarrow \mathbb{L}(\Omega\times\Omega^*,\mathcal{A})$$

given by

$$\mathcal{WV}_g\{f\}(x,\omega) := WV[f,g](x,\omega), \quad \mathcal{WV}_G\{F\}(x,\omega) := WV[F,G](x,\omega).$$

2. Generalized quantum signal/system theory on hypergroups

2.1 Basic definitions

The *basic objects of quantum harmonic analysis* (QHA) are related not to classical signals and spectra f, F but to quantum signals and quantum spectra (Hermitian operators) $\widehat{f}$, $\widehat{F}$ associated with classical signals and spectra as follows:

$$f \to \mathrm{AW}[f] \to \widehat{f}, \quad F \to \mathrm{AW}[F] \to \widehat{F}.$$

These maps are called the *Weyl quantizations* of signals and spectra, respectively. There are also the *Schwinger quantizations* using Wigner–Ville distributions:

$$f \to \mathrm{WV}[f] \to \widehat{f}, \quad F \to \mathrm{WV}[F] \to \widehat{F}.$$

The functions $\mathrm{AW}[f](\nu, v)$, $\mathrm{AW}[F](\nu, v)$ (or $\mathrm{WV}[f](x,\omega)$, $\mathrm{WV}[F](x,\omega)$) are called the *symbols* (a symbol is not a kernel) of the quantum signal $\widehat{f}$ and the quantum spectra $\widehat{F}$, respectively, and are denoted by

$$\mathrm{AW}[f](\nu, v) := sym\{\widehat{f}\}, \quad \mathrm{AW}[F](v, \nu) := sym\{\widehat{F}\},$$

or

$$\mathrm{WV}[f](x, \omega) := sym\{\widehat{f}\}, \quad \mathrm{WV}[F](\omega, x) := sym\{\widehat{F}\}.$$

Vice versa, a quantum signal $\widehat{f}$ and quantum spectra $\widehat{F}$ are called the *operators associated with a classical signal* f and classical spectrum by symbols $\mathrm{AW}[f]$, $\mathrm{AW}[F]$ (or by $\mathrm{WV}[f]$, $\mathrm{WV}[F]$), respectively, and they are denoted by

$$\widehat{f} := Op\{\mathrm{AW}[f]\}, \quad \widehat{F} := Op\{\mathrm{AW}[F]\},$$

or

$$\widehat{f} := Op\{\mathrm{WV}[f]\}, \quad \widehat{F} := Op\{\mathrm{WV}[F]\}.$$

All quantum signals $\widehat{f}$ and quantum spectra $\widehat{F}$ form the following quantum spaces:

$$\mathbf{Sig}_1 := \{\widehat{f} \mid \widehat{f} \text{ are operators acting in } \mathbb{L}_2(\Omega, \mathcal{A})\},$$

$$\mathbf{Sp}_1 := \{\widehat{F} \mid \widehat{F} \text{ are operators acting in } \mathbb{L}_2(\Omega^*, \mathcal{A})\}.$$

Let $\mathbf{Sig}_1$ and $\mathbf{Sp}_1$ be the spaces of quantum signals and quantum spectra with the following scalar products and norms:

$$\langle \widehat{f}_1 | \widehat{f}_2 \rangle := \mathbf{Tr}(\widehat{f}_1 \widehat{f}_2^{\dagger}), \quad ||\widehat{f}|| := \langle \widehat{f} | \widehat{f} \rangle = \mathbf{Tr}(\widehat{f} \widehat{f}^{\dagger}),$$

$$\langle \widehat{F}_1 | \widehat{F}_2^\dagger \rangle := \mathbf{Tr}(\widehat{F}_1 \widehat{F}_2^\dagger), \quad ||\widehat{F}|| := \langle \widehat{F} | \widehat{F} \rangle = \mathbf{Tr}(\widehat{F}\widehat{F}^\dagger),$$

where $\mathbf{Tr}(.)$ denotes the trace.

DEFINITION 9 *The spaces* $\mathbf{Sig}_1$, $\mathbf{Sp}_1$ *with the scalar products* $\langle \widehat{f}_1 | \widehat{f}_2 \rangle$, $\langle \widehat{F}_1 | \widehat{F}_2 \rangle$ *and norms* $||\widehat{f}||$, $||\widehat{F}||$ *are called the Hilbert–Liouville spaces.*

Let $\{\widehat{\varphi}_\lambda\}_{\lambda\in\Lambda}$ and $\{\widehat{\psi}_\lambda\}_{\lambda\in\Lambda}$ be two Λ-parametric families of operators, parametrized by the label $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda \subset \mathbf{R}^r$ of a subset Λ of an rD space $\mathbf{R}^r$ endowed with a suitable measure $\mu(\lambda)$. These families are called the *quora* for any subalgebras $\mathbf{Alg_1} \subset \mathbf{Sig}_1$ and $\mathbf{Alg_1^*} \subset \mathbf{Sp}_1$, if every quantum signal $\widehat{f} \in \mathbf{Alg_1}$ and quantum spectrum $\widehat{F} \in \mathbf{Alg_1^*}$ is determined by all scalar products

$$\mathrm{FT}(\lambda) = \langle \widehat{f} | \widehat{\varphi}_\lambda \rangle = \mathbf{Tr}(\widehat{f}\widehat{\varphi}_\lambda^\dagger), \ \ \mathrm{TF}(\lambda) = \langle \widehat{F} | \widehat{\psi}_\lambda \rangle = \mathbf{Tr}(\widehat{F}\widehat{\psi}_\lambda^\dagger)$$

for all $\widehat{\varphi}_\lambda$ and $\widehat{\psi}_\lambda$. The fundamental property of the quora is that any quantum signal and spectrum can be expressed as integral transforms

$$\widehat{f} = \mathcal{QF}^{-1}\{\mathrm{FT}(\lambda)\} = \int\limits_{\lambda\in\Lambda} \mathrm{FT}(\lambda)\widehat{\varphi}_\lambda d\mu(\lambda) = Op\{\mathrm{FT}(\lambda)\}, \tag{30}$$

$$\widehat{F} = \mathcal{QF}^{-1}\{\mathrm{TF}(\lambda)\} = \int\limits_{\lambda\in\Lambda} \mathrm{TF}(\lambda)\widehat{\psi}_\lambda d\mu(\lambda) = Op\{\mathrm{TF}(\lambda)\}, \tag{31}$$

where

$$\mathrm{FT}(\lambda) = \mathcal{QF}\{\widehat{f}\} = \langle \widehat{f} | \widehat{\varphi}_\lambda \rangle = \mathbf{Tr}(\widehat{f}\widehat{\varphi}_\lambda^\dagger) = sym\{\widehat{f}\}, \tag{32}$$

$$\mathrm{TF}(\lambda) = \mathcal{QF}\{\widehat{f}\} := \langle \widehat{f} | \widehat{\psi}_\lambda^\dagger \rangle = \mathbf{Tr}(\widehat{f}\widehat{\psi}_\lambda) = sym\{\widehat{F}\}. \tag{33}$$

Usually, $\mathrm{FT}(\lambda)$ and $\mathrm{TF}(\lambda)$ are Wigner–Ville distribution and Woodward ambiguity functions, respectively. Further we shall use only Woodward ambiguity functions to design quantum signals and spectra.

DEFINITION 10 *Let* $\{\widehat{\varphi}_\lambda\}_{\lambda\in\Lambda}$ *and* $\{\widehat{\psi}_\lambda\}_{\lambda\in\Lambda}$ *be two* r*-parametric families of operators. Then transforms (30)–(33) are called the abstract quantum Fourier transforms for the algebras* $\mathbf{Alg_1} \subset \mathbf{Sig}_1$ *and* $\mathbf{Alg_1}^* \subset \mathbf{Sp}_1$ *associated with two quora* $\{\widehat{\varphi}_\lambda\}_{\lambda\in\Lambda}$ *and* $\{\widehat{\psi}_\lambda\}_{\lambda\in\Lambda}$*, respectively.*

2.2 Classical Weyl quantization

It is well known that for the classical shift we have

$$\widehat{T}_x^v f(x) := f(x+v) = \sum_{k=0}^{\infty} \frac{v^k}{k!} \left(\frac{d}{dx}\right)^k f(x) = \left\{ \sum_{k=0}^{\infty} \frac{v^k}{k!} \left(\frac{d}{dx}\right)^k \right\} f(x) =$$

$$\left\{\sum_{k=0}^{\infty}\frac{(iv)^k}{k!}\left(-i\frac{d}{dx}\right)^k\right\}f(x)=\left\{\sum_{k=0}^{\infty}\frac{(iv\widehat{\mathcal{D}}_x)^k}{k!}\right\}f(x)=e^{iv\hat{\mathcal{D}}_x}f(x), \quad (34)$$

where $\widehat{\mathcal{D}}_x=-i\frac{d}{dx}$. This expression represents the decomposition of the ordinary finite shift into a series of powers of the differential operator $\frac{d}{dx}$ and is called the *infinitesimal representation* of translation shift. Analogously, we can obtain $\widehat{D}_\omega^\nu F(\omega)=F(\omega+\nu)=e^{i\nu\hat{\mathcal{D}}_\omega}F(\omega)$, where $\widehat{\mathcal{D}}_\omega=-i\frac{d}{d\omega}$. Hence, $\widehat{T}_x^v=e^{iv\hat{\mathcal{D}}_x}$, $\widehat{D}_\omega^\nu=e^{i\nu\hat{\mathcal{D}}_\omega}$. In 1932 H. WEYL proposed [23] to modify the Fourier transform formula by changing its *complex-valued harmonics* into *operator-valued harmonics.* He used the following three quora for his quantization procedures of the signal space $\mathbf{Sig}_0$:

$$\left\{\mathcal{E}_x^{[\nu,v]}=e^{i(\nu\hat{\mathcal{M}}_x+v\hat{\mathcal{D}}_x)}\right\},\ \left\{\mathcal{E}_x^{(\nu,v)}=e^{i\nu\hat{\mathcal{M}}_x}e^{iv\hat{\mathcal{D}}_x}\right\},\ \left\{\mathcal{E}_x^{(v,\nu)}=e^{iv\hat{\mathcal{D}}_x}e^{i\nu\hat{\mathcal{M}}_x}\right\}$$

associated with the classical Fourier transform, where multiplication $\widehat{\mathcal{M}}_x$ and differential $\widehat{\mathcal{D}}_x$ operators are given by

$$\widehat{\mathcal{M}}_xf(x):=xf(x),\quad \widehat{\mathcal{D}}_xf(x):=-i\frac{f(x)}{dx}.$$

Using the first quorum, H. Weyl wrote any quantum signal $\widehat{f}\in\mathbf{Sig}_1$ as

$$\widehat{f}:=\mathcal{QF}_x\Big\{\mathrm{AW}[f]\Big\}=$$

$$Op\Big\{\mathrm{AW}[f]\Big\}=\int\limits_{\nu\in\Omega^*}\int\limits_{v\in\Omega}\mathrm{AW}[f](\nu,v)e^{i[\nu\hat{\mathcal{M}}_x+v\hat{\mathcal{D}}_x]}d\mu(\nu)d\mu(v), \qquad (35)$$

where

$$\mathrm{AW}[f](\nu,v)=\mathcal{QF}_x^{-1}\{\mathrm{AW}[f]\}=Sym\{\widehat{f}\}=\mathbf{Tr}\left[\widehat{f}\ e^{-i[\nu\hat{\mathcal{M}}_x+v\hat{\mathcal{D}}_x]}\right]. \quad (36)$$

Transformations (35) and (36) are called the direct and inverse *ordinary quantum Fourier transforms* in the quantum signal space. It is natural to view maps $\mathrm{AW}[f]\longrightarrow\widehat{f}$, $\widehat{f}\longrightarrow\mathrm{AW}[f]$ as operator-valued Fourier transforms. But we can write them in the explicit form of the integral kernels. For example, for the map $\mathrm{AW}[f]\longrightarrow\widehat{f}$ the kernel $f(x,y)$ of the operator $\widehat{f}$ has the form:

$$f(x,y)=\int\limits_{\nu\in\Omega^*}\mathrm{AW}[f](\nu,y-x)e^{i\nu x}e^{i\frac{\nu}{2}(y-x)}d\mu(\nu).$$

Of course, for quantization of the spectral space $\mathbf{Sp}_0$ one can use three dual quora

$$\left\{\mathcal{E}_\omega^{[v,\nu]}=e^{i[v\hat{\mathcal{M}}_\omega+\nu\hat{\mathcal{D}}_\omega]}\right\},\ \left\{\mathcal{E}_\omega^{(v,\nu)}=e^{iv\hat{\mathcal{M}}_\omega}e^{i\nu\hat{\mathcal{D}}_\omega}\right\},\ \left\{\mathcal{E}_\omega^{(v,\nu)}=e^{i\nu\hat{\mathcal{D}}_\omega}e^{iv\hat{\mathcal{M}}_\omega}\right\},$$

where $\widehat{\mathcal{M}}_\omega F(\omega) := \omega F(\omega)$, $\widehat{\mathcal{D}}_\omega F(\omega) := -i\frac{F(\omega)}{d\omega}$. Using the first quorum, we can write any quantum spectrum $\widehat{F} \in \mathbf{Sp}_1$ as follows:

$$\widehat{F} := \mathfrak{Q}\mathcal{F}_\omega\Big\{\mathrm{AW}[F]\Big\} =$$

$$Op\Big\{\mathrm{AW}[F]\Big\} = \int\limits_{v\in\Omega}\int\limits_{\nu\in\Omega^*} \mathrm{AW}[F](v,\nu)e^{i[v\hat{\mathcal{M}}_\omega+\nu\hat{\mathcal{D}}_\omega]}d\mu(v)d\mu(\nu), \qquad (37)$$

where

$$\mathrm{AW}[F](v,\nu) = \mathfrak{Q}\mathcal{F}_x^{-1}\{\mathrm{AW}[F]\} = Sym\{\widehat{F}\} = \mathbf{Tr}\left[\widehat{F}\ e^{-i[v\hat{\mathcal{M}}_\omega+\nu\hat{\mathcal{D}}_\omega]}\right], \qquad (38)$$

Transformations (37) and (38) are called the direct and inverse *ordinary quantum Fourier transforms* in the quantum spectral space.

2.3 Generalized Heisenberg–Weyl operators

Let us construct generalized operator-valued hyperharmonics associated with an orthogonal basis $\{\varphi_\omega(x)\}_{\omega\in\Omega^*}$.

Definition 11 *Operator $\widehat{\mathcal{D}}_x$, for which $\widehat{\mathcal{D}}_x\varphi_\omega(x) = \omega\varphi_\omega(x)$ is valid, is called the generalized differential operator.*

The generalized differential operator appears as an ordinary differential operator with variable coefficients, for example $\widehat{\mathcal{D}}_x = p_2(x)\frac{\widehat{D}^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x)$, where $p_2(x)$, $p_1(x)$, $p_0(x)$ are some variable coefficients. Let us now find a connection between the GSOs $\widehat{T}_x^v$ and the generalized differential operator $\widehat{\mathcal{D}}_x$. It can be found using the Taylor expansion.

Theorem 3 *Let $\{\varphi_\omega(x)\}_{\omega\in\Omega^*} \in \mathbf{Sig}_0$ be some Fourier basis, consisting of $\mathcal{A}$-valued basis functions. Then all GSOs associated with it have the infinitesimal representation: $\widehat{T}_x^v = \varphi_{\hat{\mathcal{D}}_x}(v)$, $\widehat{D}_\omega^\nu = \varphi_\nu(\widehat{\mathcal{D}}_\omega)$, and are called the operator-valued hyperharmonics associated with an orthogonal basis $\{\varphi_\omega(x)\}_{\omega\in\Omega^*}$, where $\widehat{\mathcal{D}}_x\varphi_\omega(x) = \omega\varphi_\omega(x)$, $\widehat{\mathcal{D}}_\omega\varphi_\omega(x) = x\varphi_\omega(x)$.*

Proof: If the signals $\varphi_\omega(v)$ are decomposed into the following series of ω, $\varphi_\omega(v) = \sum_{k=0}^\infty X_k(v)(\omega)^k$, then we can construct the operators $\varphi_{\hat{\mathcal{D}}_x}(v) = \sum\limits_{k=0}^\infty X_k(v)\widehat{\mathcal{D}}_x^k$. For these operators we have

$$\Big(\varphi_{\hat{\mathcal{D}}_x}(v)\Big)\varphi_\omega(x) = \left(\sum_{k=0}^\infty X_k(v)\widehat{\mathcal{D}}_x^k\right)\varphi_\omega(x) = \left(\sum_{k=0}^\infty X_k(v)(\omega)^k\right)\varphi_\omega(x) =$$

$$\varphi_\omega(v)\varphi_\omega(x) = \varphi_\omega(x \boxplus v) = \widehat{T}_x^v \varphi_\omega(x), \tag{39}$$

i.e., $\widehat{T}_x^v = \varphi_{\hat{\mathcal{D}}_x}(v)$. Analogously, $\widehat{D}_\omega^\nu = \varphi_\nu(\widehat{\mathcal{D}}_\omega)$. Obviously, $M_x^\nu = \varphi_\nu(\hat{\mathcal{M}}_x)$ and $M_\omega^v = \varphi_{\widehat{\mathcal{M}}_\omega}(v)$. □

Using the hyperharmonics $\widehat{T}_x^v = \varphi_{\hat{\mathcal{D}}_x}(v)$ and $\widehat{D}_\omega^\nu = \varphi_\nu(\widehat{\mathcal{D}}_\omega)$ associated with the basis $\{\varphi_\omega(x)\}_{\omega\in\Omega^*}$, we can construct generalized Heisenberg–Weyl operators and quantum hyperharmonic analysis of quantum signals and spectra.

The "time"-shift and "frequency"-shift operators together acting on spaces $\mathbf{Sig}_0$ and $\mathbf{Sp}_0$ induce the following pair of sets of the *Heisenberg–Weyl operators*:

$$\mathbb{HW}_x := \left\{\widehat{\mathcal{E}}_x^{(\nu,v)} = \widehat{M}_x^\nu \widehat{T}_x^v = \varphi_\nu(\hat{\mathcal{M}}_x)\varphi_{\hat{\mathcal{D}}_x}(v) \mid \nu\in\Omega^*,\ v\in\Omega\right\},$$

$$\mathbb{HW}_\omega := \left\{\widehat{\mathcal{E}}_\omega^{(v,\nu)} = \widehat{M}_\omega^v \widehat{D}_\omega^\nu = \varphi_{\widehat{\mathcal{M}}_\omega}(v)\varphi_\nu(\widehat{\mathcal{D}}_\omega) \mid \nu\in\Omega^*,\ v\in\Omega\right\}.$$

They act on $\mathbf{Sig}_0$ and $\mathbf{Sp}_0$ by the following rules:

$$\widehat{\mathcal{E}}_x^{(\nu,v)} f(x) := \left(\widehat{M}_x^\nu \widehat{T}_x^v f\right)(x) = \varphi_\nu(x) f(x \boxplus v),$$

$$\widehat{\mathcal{E}}_\omega^{(v,\nu)} F(\omega) := \left(\widehat{M}_\omega^v \widehat{D}_\omega^\nu F\right)(\omega) = \varphi_\omega(v) F(\omega \oplus \nu).$$

Obviously,

$$\underset{x\to\omega}{\mathcal{F}_0}\left\{\widehat{\mathcal{E}}_x^{(\nu,v)} f(x)\right\} = \overline{\varphi}_\nu(v)\widehat{\mathcal{E}}_\omega^{(v,-\nu)} F(\omega)$$

and

$$\underset{\omega\to x}{\mathcal{F}_0^{-1}}\left\{\widehat{\mathcal{E}}_\omega^{(v,\nu)} F(\omega)\right\} = \varphi_\nu(v)\widehat{\mathcal{E}}_x^{(\nu,v)} f(x).$$

Now we construct two sets of symmetric Heisenberg–Weyl operators:

$$\mathbb{SHW}_x = \left\{\widehat{\mathcal{E}}_x^{[\nu,v]} = \varphi_\nu^{1/2}(v)\varphi_\nu(\hat{\mathcal{M}}_x)\varphi_{\hat{\mathcal{D}}_x}(v)\,\middle|\,\nu\in\Omega^*, v\in\Omega\right\},$$

$$\mathbb{SHW}_\omega = \left\{\widehat{\mathcal{E}}_\omega^{[v,\nu]} = \overline{\varphi}_\nu^{1/2}(v)\varphi_{\widehat{\mathcal{M}}_\omega}(v)\varphi_\nu(\widehat{\mathcal{D}}_\omega)\,\middle|\,\nu\in\Omega^*, v\in\Omega\right\}.$$

These operators satisfy the following composition laws:

$$\widehat{\mathcal{E}}_x^{[\nu,v]}\widehat{\mathcal{E}}_x^{[\nu',v']} = \overline{\varphi}_\nu^{1/2}(v')\varphi_{\nu'}^{1/2}(v)\widehat{\mathcal{E}}_x^{[\nu+\nu',v+v']}$$

$$\widehat{\mathcal{E}}_\omega^{[v,\nu]}\widehat{\mathcal{E}}_\omega^{[v',\nu']} = \varphi_\nu^{1/2}(v')\overline{\varphi}_{\nu'}^{1/2}(v)\widehat{\mathcal{E}}_\omega^{[v\oplus v',\nu\oplus\nu']}$$

and the "commutation" relations

$$\widehat{\mathcal{E}}_x^{[\nu,v]}\widehat{\mathcal{E}}_x^{[\nu',v']} = \overline{\varphi}_\nu(v')\varphi_{\nu'}(v)\widehat{\mathcal{E}}_x^{[\nu',v']}\widehat{\mathcal{E}}_x^{[\nu,v]},$$

$$\widehat{\mathcal{E}}_\omega^{[v,\nu]}\widehat{\mathcal{E}}_\omega^{[v',\nu']} = \varphi_\nu(v')\overline{\varphi}_{\nu'}(v)\widehat{\mathcal{E}}_\omega^{[v',\nu']}\widehat{\mathcal{E}}_\omega^{[v,\nu]}.$$

2.4 Generalized Weyl quantizations

Let us consider the linear quantum spaces $\mathbf{Sig}_1$ and $\mathbf{Sp}_1$ of quantum signals $\widehat{f}$ and quantum spectra $\widehat{F}$, respectively. The inner product can be defined by $\langle \widehat{f}_1|\widehat{f}_2\rangle := \mathbf{Tr}(\widehat{f}_1\widehat{f}_2^\dagger)$, $\quad \langle \widehat{F}_1|\widehat{F}_2\rangle := \mathbf{Tr}(\widehat{F}_1\widehat{F}_2^\dagger)$. It is easy to check that

$$\mathbf{Tr}\left[\widehat{\mathcal{E}}_x^{[\nu,v]}\left(\widehat{\mathcal{E}}_x^{[\nu',v']}\right)^\dagger\right] = \delta(\nu \boxminus \nu')\delta(v \boxminus v'), \tag{40}$$

$$\mathbf{Tr}\left[\widehat{\mathcal{E}}_\omega^{[v,\nu]}\left(\widehat{\mathcal{E}}_\omega^{[v',\nu']}\right)^\dagger\right] = \delta(v \ominus v')\delta(\nu \ominus \nu'). \tag{41}$$

The families $\left\{\widehat{\mathcal{E}}_x^{[\nu,v]}\right\}_{[\nu,v]\in\Omega^*\times\Omega}$ and $\left\{\widehat{\mathcal{E}}_\omega^{[v,\nu]}\right\}_{[v,\nu]\in\Omega\times\Omega^*}$ form two quora in quantum spaces. For this reason, any quantum signal $\widehat{f} \in \mathbf{Sig}_1$ and quantum spectra $\widehat{F} \in \mathbf{Sp}_1$ can be written as follows:

$$\widehat{f} = \mathfrak{QF}_x\Big\{\mathrm{AW}[f]\Big\} = Op\Big\{\mathrm{AW}[f]\Big\} = \int\limits_{\nu\in\Omega^*}\int\limits_{v\in\Omega}\mathrm{AW}[f](\nu,v)\widehat{\mathcal{E}}_x^{[\nu,v]}d\mu(\nu)d\mu(v), \tag{42}$$

$$\begin{aligned}\widehat{F} &= \mathfrak{QF}_\omega\Big\{\mathrm{AW}[F]\Big\} \\ &= Op\Big\{\mathrm{AW}[f]\Big\} \\ &= \int\limits_{v\in\Omega}\int\limits_{\nu\in\Omega^*}\mathrm{AW}[F](v,\nu)\widehat{\mathcal{E}}_\omega^{[v,\nu]}\,d\mu(v)\,d\mu(\nu).\end{aligned} \tag{43}$$

Using (40) and (41), one can invert (42) and (43) as follows:

$$\mathrm{AW}[f](\nu,v) = \mathfrak{QF}_x^{-1}\{\mathrm{AW}[f]\} = Sym\{\widehat{f}\} = \mathbf{Tr}\left[\widehat{f}\left(\widehat{\mathcal{E}}_x^{[\nu,v]}\right)^\dagger\right], \tag{44}$$

$$\mathrm{AW}[F](v,\nu) = \mathfrak{QF}_x^{-1}\{\mathrm{AW}[F]\} = Sym\{\widehat{F}\} = \mathbf{Tr}\left[\widehat{F}\left(\widehat{\mathcal{E}}_\omega^{[v,\nu]}\right)^\dagger\right]. \tag{45}$$

The transformations (42) and (45) are called the *generalized quantum Fourier transforms.*

EXAMPLE 5 *In this example we consider the Weyl quantization on a finite cyclic group $\Omega = \Omega^* = \mathbb{Z}/p$, where p is a prime integer. In this case,*

$$\widehat{\mathcal{E}}_x^{[\nu,v]} = \varepsilon^{\frac{\nu v}{2}}\widehat{\mathcal{E}}_x^{(\nu,v)} = \varepsilon^{\frac{\nu v}{2}}\widehat{M}_x^\nu\widehat{T}_x^v =$$

$$= \varepsilon^{\frac{\nu v}{2}} \begin{bmatrix} 1 & & & & \\ & \varepsilon^1 & & & \\ & & \varepsilon^2 & & \\ & & & \ddots & \\ & & & & \varepsilon^{p-1} \end{bmatrix}^{\nu} \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}^{v}.$$

For this reason, the map

$$\widehat{f} = \mathfrak{QF}_x\Big\{AW[f]\Big\} = Op\Big\{AW[f]\Big\} = \sum_{\nu\in\mathbb{Z}/p}\sum_{v\in\mathbb{Z}/p} AW[f](\nu,v)\widehat{\mathcal{E}}_x^{[\nu,v]} =$$

$$= \sum_{\nu\in\mathbb{Z}/p}\sum_{v\in\mathbb{Z}/p} AW[f](\nu,v)\varepsilon^{\frac{\nu v}{2}} \begin{bmatrix} 1 & & & & \\ & \varepsilon^1 & & & \\ & & \varepsilon^2 & & \\ & & & \ddots & \\ & & & & \varepsilon^{p-1} \end{bmatrix}^{\nu} \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}^{v} \tag{46}$$

is the discrete quantum Fourier transform associated with the cyclic group $\mathbb{Z}$.

2.5 Generalized quantum convolutions

For the product of two quantum signals $\widehat{f}$ and $\widehat{g}$ we have

$$\widehat{f}\widehat{g} = \int\limits_{(\nu,v)}\int\limits_{(\nu',v')} \mathrm{AW}[f](\nu,v)\widehat{\mathcal{E}}_x^{[\nu,v]}\mathrm{AW}[g](\nu',v')\widehat{\mathcal{E}}_x^{[\nu',v']}d\mu(\nu,v)d\mu(\nu',dv') =$$

$$\int\limits_{(\omega,x)} \Big(\mathrm{AW}[f]\circledast\mathrm{AW}[g]\Big)(\omega,x)\widehat{\mathcal{E}}_x^{[\omega,x]}d\mu(\omega)dx =$$

$$\mathfrak{QF}_x\left\{\mathrm{AW}[f]\circledast\mathrm{AW}[g]\right\} = Op\left\{\mathrm{AW}[f]\circledast\mathrm{AW}[g]\right\},$$

where the expression

$$\Big(\mathrm{AW}[f]\circledast\mathrm{AW}[g]\Big)(\omega,x) = \mathfrak{QF}_x^{-1}\{\widehat{f}\widehat{g}\} = sym\{\widehat{f}\widehat{g}\} =$$

$$\int\limits_{(\nu,v)} \mathrm{FT}(\nu,v)\mathrm{GT}(\omega\ominus\nu, x\boxminus v)\bar{\varphi}_{\nu'}^{1/2}(v)\varphi_{\nu}^{1/2}(v')d\mu(\nu)dv \tag{47}$$

is called the *generalized twisted signal convolution.* Analogously,

$$\widehat{F}\widehat{G} = \int\limits_{(v,\nu)}\int\limits_{(v',\nu')} \mathrm{AW}[F](v,\nu)\widehat{\mathcal{E}}_\omega^{[v,\nu]}\mathrm{AW}[F](\nu',v')\widehat{\mathcal{E}}_\omega^{[v',\nu']}d\mu(\nu)dv\ d\mu(\nu')dv' =$$

$$\int\limits_{(x,\omega)} \Big(\mathrm{AW}[F]\bigstar\mathrm{AW}[G]\Big)(x,\omega)\widehat{\mathcal{E}}_{\omega}^{[x,\omega]}dx\ d\mu(\omega) =$$

$$\mathfrak{QF}_{\omega}\left\{\mathrm{AW}[F]\bigstar\mathrm{AW}[G]\right\} = Op\left\{\mathrm{AW}[F]\bigstar\mathrm{AW}[G]\right\},$$

where

$$\left(\mathrm{AW}[F]\bigstar\mathrm{AW}[G]\right)(x,\omega) := \mathfrak{QF}_{\omega}^{-1}\{\widehat{F}\widehat{G}\} = sym\left\{\widehat{F}\widehat{G}\right\} =$$

$$\int\limits_{(v,\nu)} \mathrm{AW}[F](v,\nu)\mathrm{AW}[G](x\boxminus v,\omega\ominus\nu)\varphi_{\nu'}^{1/2}(v)\bar{\varphi}_{\nu}^{1/2}(v')dv\ d\mu(\nu) \qquad (48)$$

is called the *generalized twisted spectral convolution.*

According to the Pontryagin duality principle we can define the *generalized quantum convolution* of quantum signals by

$$\widehat{f}\circledast\widehat{g} := Op\{\mathrm{AW}[f]\mathrm{AW}[g]\} = \mathfrak{QF}_x^{(s)}\{\mathrm{AW}[f]\mathrm{AW}[g]\},$$

where

$$\mathrm{AW}[f](\nu,v)\mathrm{AW}[g][\nu,v] =$$

$$sym\{\widehat{f}\underset{1}{\circledast}\widehat{g}\} = \mathfrak{QF}_x^{-1}\{\widehat{f}\underset{1}{\circledast}\widehat{g}\} = \mathbf{Tr}\left[\left(\widehat{f}\circledast\widehat{g}\right)\left(\mathcal{E}_x^{[\nu,v]}\right)^{\dagger}\right],$$

and the *generalized quantum convolution* of quantum spectra by

$$\widehat{F}\bigstar\widehat{G} := Op\Big\{\mathrm{AW}[F]\mathrm{AW}[G]\Big\} = \mathfrak{QF}_{\omega}\Big\{\mathrm{AW}[F]\mathrm{AW}[G]\Big\},$$

where

$$\mathrm{AW}[F](\nu,v)\mathrm{AW}[G][\nu,v] =$$

$$sym\left\{\widehat{F}\bigstar\widehat{G}\right\} = \mathfrak{QF}_x^{-1}\left\{\widehat{F}\underset{1}{\bigstar}\widehat{G}\right\} = \mathbf{Tr}\left[\left(\widehat{F}\bigstar\widehat{G}\right)\left(\mathcal{E}_x^{[\nu,v]}\right)^{\dagger}\right].$$

THEOREM 4 *The quantum generalized convolutions and quantum generalized Fourier transforms are related by the expressions:*

$$\mathfrak{QF}_x\left\{AW[f]\circledast AW[g]\right\} = \widehat{f}\widehat{g}, \quad \mathfrak{QF}_x\left\{AW[F]\bigstar AW[G]\right\} = \widehat{F}\widehat{G},$$

and

$$\mathfrak{QF}_x^{-1}\left\{\widehat{f}\circledast\widehat{g}\right\} = AW[f](\nu,v)AW[g][\nu,v],$$

$$\mathfrak{QF}_x^{-1}\left\{\widehat{F}\bigstar\widehat{G}\right\} = AW[F](\nu,v)AW[G][\nu,v].$$

3. Conclusion

In this paper we have examined the idea of generalized shift operators associated with an arbitrary orthogonal transform and generalized linear and nonlinear convolutions based on these generalized shift operators. Such operators allow one to unify and generalize the majority of known methods and tools of signal processing based on the classical Fourier transform for generalized classical and quantum signal theories.

References

[1] Marty, F. (1934). Sur une generalization de la notion de groupe. *Sartryck ur Forhandlingar via Altonde Skandinavioka Matematiker kongressen i Stockholm,* pp. 45–49.

[2] Marty, F. (1935). Role de la notion d'hypergroupe dans l'etude des groupes non abelians. *Computes Rendus de l'Academie des Sciences,* **201**, pp. 636–638.

[3] Wall, H.S. (1934). Hypergroups. *Bulletin of the American Mathematical Socity,* **41**, 36–40 [Presented at the annual meeting of the American Mathematical Society, Pittsburgh, December 27–31, 1934].

[4] Wall, H.S. (1937). Hypergroups. *American Journal of Mathematics.* **59**, pp. 77–98.

[5] Levitan, B.M. (1949). The application of generalized displacement operators to linear differential equations of second order. *Uspechi Math. Nauk,* **4**, No 1(29), pp. 3–112 (English transl., *Amer. Mat. Soc. Transl.* (1), **10**, 1962, 408–541, MR 11, 116).

[6] Levitan, B.M. (1964). Generalized translation operators.*Israel Program for Scientific Translations,* Jerusalem, 120 p.

[7] Dunkl, C.F. (1966). Operators and harmonic analysis on the sphere. *Trans. Amer. Math. Soc.* **125**, 2, pp. 50–263.

[8] Jewett, R.I. (1975). Spaces with an abstract convolution of measures, *Advances in Math.*, **18**, pp. 1–101.

[9] Labunets, V.G., Sitnikov, O.P. (1976). Generalized harmonic analysis of **VP**-invariant systems and random processes. In: *Harmonic Analysis on Groups in Abstract Systems Theory.* (in Russian), Ural State Technical University, Sverdlovsk, pp. 44–67.

[10] Labunets, V.G. and Sitnikov, O.P. (1976). Generalized harmonic analysis of VP-invariant linear sequential circuits. In: *Harmonic Analysis on Groups in Abstract System Theory (in Russian).* Ural Polytechnical Institute Press: Sverdlovsk, pp. 67–83.

[11] Labunets-Rundblad, E.V., Labunets, V.G., and Astola, J. (2000). Algebraic frames for commutative hyperharmonic analysis of signals and images. *Algebraic Frames for the Perception-Action Cycle.* Second Inter. Workshop, AFPAC 2000, Kiel, Germany, September 2000. Lectures Notes in Computer Science, 1888, Berlin, 2000, pp. 294–308.

[12] Creutzburg, R., Labunets, E.V., and Labunets, V.G. (1998). Algebraic foundations of an abstract harmonic analysis of signals and systems. *Workshop on*

Transforms and Filter Banks, Tampere International Center for Signal Processing, pp. 30–68.

[13] Creutzburg, R., Labunets, E., and Labunets V. (1992) Towards an "Erlangen program" for general linear system theory. Part I. In: F. Pichler (Edt), *Lecture Notes in Computer Science,* Vol. 585, Springer: Berlin, pp. 32–51.

[14] Creutzburg, R., Labunets, E., and Labunets, V. (1994). Towards an "Erlangen program" for general linear systems theory. Part II. *Proceed. EUROCAST'93* (Las Palmas, Spain) F. Pichler, R. Moreno Diaz (Eds.) In: *Lecture Notes in Computer Science*, Vol. 763, Springer: Berlin, pp. 52–71.

[15] Labunets, V.G. (1993). Relativity of "space" and "time" notions in system theory. In: *Orthogonal Methods Application to Signal Processing and Systems Analysis* (in Russian). Ural Polytechnical Institute Press: Sverdlovsk, pp. 31–44.

[16] Labunets, V.G. (1982). Spectral analysis of linear dynamic systems invariant with respect to generalized shift operators. In: *Experimental Investigations in Automation* (in Russian). Institute of Technical Cybernetics of Belorussian Academy of Sciences Press: Minsk, pp. 33–45.

[17] Labunets, V.G. (1982). Algebraic approach to signals and systems theory: linear systems examples. In: *Radiioelectronics Devices and Computational Technics Means Design Automation* (in Russian). Ural Polytechnical Institute Press: Sverdlovsk, pp. 18–29.

[18] Labunets, V.G. (1980). Examples of linear dynamical systems invariant with respect to generalized shift operators. In: *Orthogonal Methods for the Application to Signal Processing and Systems Analysis* (in Russian): Ural Polytechnical Institute Press: Sverdlovsk, pp. 4–14.

[19] Labunets, V. G. (1980). Symmetry principles in the signal and system theories. In: *Synthesis of Control and Computation Systems* (in Russian). Ural Polytechnical Institute Press: Sverdlovsk, pp. 14–24.

[20] Wigner, E.R. (1932). On the quantum correction for thermo-dynamic equilibrium. *Physics Review,* **40**, pp. 749–759.

[21] Ville, J. (1948). Theorie et Applications de la Notion de Signal Analytique. *Gables et Transmission.* **2A**, pp. 61–74.

[22] Woodward, P.M. (1951). Information theory and design of radar receivers. *Proceedings of the Institute of Radio Engineers,* **39**, pp. 1521–1524.

[23] Weyl, H. (1932) *The Theory Group and Quantum Mechanics,* London: Methuen, 236 p.