A UNIFIED APPROACH TO FOURIER-CLIFFORD-PROMETHEUS SEQUENCES, TRANSFORMS AND FILTER BANKS

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Abstract In this paper we develop a new unified approach to the so-called generalized *Fourier-Clifford-Prometheus sequences*, *transforms* (FCPTs) and *M-channel Filter Banks*. It is based on a new generalized FCPT-generating construction. This construction has a rich algebraic structure that supports a wide range of fast algorithms.

Keywords: Clifford algebra, filter banks, Golay-Shapiro sequences, Fourier-Clifford-Prometheus transforms.

1. Introduction

The basis which has come to be known as the *Prometheus Orthonormal Set* (PONS) was introduced in [1] to prove the H.S. Shapiro global uncertainty principle conjecture. Each function in PONS is called a *Golay-Shapiro sequence*. They are defined on $[0, 1]$, piecewise ± 1 and can change sign only at points of the form $j/2^n$, $j = 0, 1, \ldots, 2^n - 1$, $n = 1, 2, \ldots$. These basis functions satisfy almost all standard properties of the Walsh functions. Discrete classical Fourier-Prometheus Transforms (FPT) in bases of different Golay-Shapiro sequences can be used in many signal processing applications: multiresolution by discrete orthogonal wavelet decomposition, digital audio, digital video broadcasting, communication systems (Orthogonal Frequency Division Multiplexing, Multi-Code Code-Division Multiple Access), radar, and cryptographic systems.

Golay-Shapiro (GS) 2-complementary (± 1)-valued sequences associated with the cyclic group $\mathbf{Z}_2$ were introduced by Shapiro and Golay in 1949–1951 [2]–[7]. In 1961, Golay [3] gave an explicit construction for binary Golay complementary pairs of length 2^m and later noted [4] that the construction implies the existence of at least $2^m m!/2$ binary Golay sequences of this length. They are known to exist for all lengths $N = 2^\alpha 10^\beta 26^\gamma$, where α, β, γ are integers and $\alpha, \beta, \gamma \geq 0$ [8], but do not exist for any length N having a prime factor congruent to 3 modulo 4 [9]. Budisin [10] using the earlier work of Sivaswamy [11], gave a more general recursive construction for Golay complementary pairs and showed that the set of all binary Golay complementary pairs of length 2^m obtainable from it concides with those given explicitly by Golay [3]. For a survey of results on nonbinary Golay Complementary pairs, see [12]–[13]. Recently, Davis and Jedwab [14], combining results appearing in the work of Golay and Shapiro cited above, gave an explicit description of a large class of Golay complementary sequences in terms of certain cosets of the first order Reed-Muller codes. The following general elements are used for building the classical Fourier-Prometheus transforms in bases of classical Golay-Shapiro sequences: 1) the Abelian group $\mathbf{Z}_2^n$, 2) the 2-point Fourier transform $\mathcal{F}_2$, and 3) the complex field $\mathbf{C}$; i.e., these transforms are associated with the triple $(\mathbf{Z}_2^n, \mathcal{F}_2, \mathbf{C})$.

The multiresolution analysis (MRA) operates upon a discrete signal $x(l)$ of length 2^n, where n is an integer. The sequence $x(l)$ is convolved with two filters L and H. Each convolution results in a sequence half the length of the original sequence. The result from the convolution with the low-pass filter is again transformed. Each re-transformed sequence of the low-pass output is referred to as a dilation. For a sequence $x(l)$ of length 2^n, a maximum of n dilations can be performed. MRA applied to a real-valued sequence $x(l)$ is defined recursively by the equations:

$$c^{(p)}(l) = L\left\{c^{(p-1)}(l)\right\}, \quad d^{(p)} = H\left\{c^{(p-1)(l)}\right\},$$

where $p = n, n-1, \ldots, 1, 0$, $c^n(l) = x(l)$, and

$$c(l) = (Lx)(l) = \sum_{l=0}^{2^n-1} k_{lp}(l-2k)x(l),$$

$$d(l) = (Hx)(l) = \sum_{l=0}^{2^n-1} k_{hp}(l-2k)x(l)$$

are low-pass and high-pass filters, respectively.

The sequences $c^{(p)}(l)$ and $d^{(p)}(l)$ are called the "averages" and "differences" of the original signal. The inverse discrete wavelet transform reconstructs $c^{(n)}(l) = x(l)$ using the recursive algorithm

$$c^{(p+1)}(l) = L^*\{c^{(p)}(l)\} + H^*\{d^{(p)}(l)\},$$

where L^* and H^* are the inverse filters of L and H, respectively. All filters L, H, and L^*, H^*, satisfy the following equation $LL^* = I, \quad HH^* = I$, and

$$LL^* + HH^* = 2I, \quad LH^* = H^*L = 0, \tag{1}$$

where I and 0 denote the identity and zero operators. Note that a pair of filters having these properties required of the transformations L and H are known as quadrature mirror filters, having the perfect reconstruction property.

The conditions (1) can be rewritten in terms of the $\mathcal{Z}$-transform as

$$|k_{lp}(z)|^2 + |k_{hp}(z)|^2 = 2, \quad \overline{k}_{lp}(z)k_{lp}(-z) + \overline{k}_{hp}(z)k_{hp}(-z) = 0, \quad \forall z \in \mathbb{T}_1$$

where $\mathbb{T}_1$ is the unit circle of the complex field $\mathbb{C}$. These conditions mean that impulse responses $k_{lp}(l)$ and $k_{hp}(l)$ form a Golay-Shapiro (GS) 2-complementary pair.

In this paper we develop a new unified approach to the so-called generalized *Fourier-Clifford-Prometheus* (FCP) *sequences*, FCP *transforms* (FCPTs), and *M-channel Filter Banks*. We describe the precise theoretical and computational relationship between M-band wavelets, M-channel filterbanks and generalized Golay-Shapiro sequences. The approach is based on a new generalized FCPT-generating construction. This construction has a rich algebraic structure that supports a wide range of fast algorithms. This construction is associated not with the triple $\left(\mathbf{Z}_2^n, \mathcal{F}_2, \mathbf{C}\right)$, but rather with other groups instead of $\mathbf{Z}_2^n$, other unitary transforms instead of $\mathcal{F}_2$, and other algebras (Clifford algebras) instead of the complex field $\mathbf{C}$.

2. New construction of classical and multiparametric Prometheus transforms

We begin by describing the original Golay 2-complementary (± 1)-valued sequences.

Definition 1 *Let* $\mathbf{p}(t) := (p_0, p_1, \ldots, p_{N-1})$, $\mathbf{q}(t) := (q_0, q_1, \ldots, q_{N-1})$, *where* $p_i, q_i \in \{\pm 1\}$. *The sequences* $\mathbf{p}(t), \mathbf{q}(t)$ *are called a 2-complementary* (± 1)*-valued or Golay complementary pair over* $\{\pm 1\}$ *if*

$$\mathrm{COR}[\mathbf{p}, \mathbf{p}](\tau) + \mathrm{COR}[\mathbf{q}, \mathbf{q}](\tau) = N\delta(\tau),$$

or

$$|\mathbf{p}(z)|^2 + |\mathbf{q}(z)|^2 = N, \quad \forall z \in \mathbb{T}_1,$$

where $\mathrm{COR}[\mathbf{f}, \mathbf{f}](\tau)$ *is the periodic correlation function of* $\mathbf{f}(t)$; $\mathbf{p}(z)$ *and* $\mathbf{q}(z)$ *are* $\mathcal{Z}$*-transforms of* $\mathbf{p}(t)$ *and* $\mathbf{q}(t)$, *respectively. Any sequence which is a member of a Golay complementary pair is called a Golay sequence.*

The Fourier-Prometheus matrix of depth n has size $2^n \times 2^n$: $\mathcal{FP}_{2^n} = [\mathrm{Pr}_\alpha(t)]_{\alpha,t=0}^{2^n-1}$. For α and t we shall use binary representations $\alpha = \alpha_{[n]} := (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $t = t_{[n]} := (t_1, t_2, \ldots, t_n)$, where $\alpha_i, t_i \in \{0,1\}$, $i = 1, 2, \ldots, n$. Obviously, $\alpha_{[1]} = (\alpha_1)$, $\alpha_{[2]} = (\alpha_1, \alpha_2)$, $\alpha_{[3]} = (\alpha_1, \alpha_2, \alpha_3)$, $\ldots$ $t_{[1]} = (t_1)$, $t_{[2]} = (t_1, t_2)$, $t_{[n]} = (t_1, t_2, \ldots, t_n), \ldots$. For this reason,

$$
{}^{2^n}\mathcal{FP}_{(\alpha_{[n-1]},\alpha_n)} = \left[\begin{array}{l}
\mathrm{Pr}_{(0,0,\ldots,0,0)}(t_1,\ldots,t_n) \\
\mathrm{Pr}_{(0,0,\ldots,0,1)}(t_1,\ldots,t_n) \\ \hline
\mathrm{Pr}_{(0,0,\ldots,1,0)}(t_1,\ldots,t_n) \\
\mathrm{Pr}_{(0,0,\ldots,1,1)}(t_1,\ldots,t_n) \\ \hline
\cdots \\
\cdots \\ \hline
\mathrm{Pr}_{(1,1,\ldots,1,0}(t_1,\ldots,t_n) \\
\mathrm{Pr}_{(1,1,\ldots,1,1}(t_1,\ldots,t_n)
\end{array}\right] = \boxplus_{\alpha_{[n-1]}=0}^{2^{n-1}-1} \begin{bmatrix} \mathrm{Pr}_{(\alpha_{[n-1]},0)}(t) \\ \mathrm{Pr}_{(\alpha_{[n-1]},1)}(t) \end{bmatrix},
$$

where $\mathrm{Pr}_{(\alpha_{[n-1]},0)}(t)$ and $\mathrm{Pr}_{(\alpha_{[n-1]},1)}(t)$ are a pair of GS 2-complementary sequences and $\boxplus$ represents the vertical concatenation of matrices.

The classical matrix $\mathcal{FP}_{2^n}$ is formed by starting with the (2×2)-matrix ${}^{2^1}\mathcal{FP} = \begin{bmatrix} \mathrm{Pr}_0(t) \\ \mathrm{Pr}_1(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and by repeated application of the **PONS**-iteration construction to pairs of rows in the matrix. In the $(n+1)$st iteration this construction takes each pair $\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathrm{Pr}_{(\alpha_{[n-1]},0)}(t) \\ \mathrm{Pr}_{(\alpha_{[n-1]},1)}(t) \end{bmatrix}$ of

$$
{}^{2^n}\mathcal{FP}_{(\alpha_{[n-1]},\alpha_n)} = \boxplus_{\alpha_{[n-1]}=0}^{2^{n-1}-1} \begin{bmatrix} \mathrm{Pr}_{(\alpha_{[n-1]},0)}(t) \\ \mathrm{Pr}_{(\alpha_{[n-1]},1)}(t) \end{bmatrix}
$$

and constructs four rows of twice the length

$$
\mathbf{PONS}(\mathbf{p},\mathbf{q}) = \left[\begin{array}{rr}
\mathbf{p} & \mathbf{q} \\
\mathbf{p} & -\mathbf{q} \\ \hline
\mathbf{q} & \mathbf{p} \\
-\mathbf{q} & -\mathbf{p}
\end{array}\right] = \begin{bmatrix} \mathbf{p} & \mathbf{q} \\ \mathbf{p} & -\mathbf{q} \end{bmatrix} \boxplus \begin{bmatrix} \mathbf{q} & \mathbf{p} \\ -\mathbf{q} & -\mathbf{p} \end{bmatrix}
$$

$$
= \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left[\begin{array}{c|c} \mathbf{p} & \\ \hline & \mathbf{q} \end{array}\right] \right) \boxplus \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left[\begin{array}{c|c} & \mathbf{p} \\ \hline \mathbf{q} & \end{array}\right] \right)
$$

$$
= \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left[\begin{array}{c|c} \mathbf{p} & \\ \hline & \mathbf{q} \end{array}\right] \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \right) \boxplus \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left[\begin{array}{c|c} \mathbf{p} & \\ \hline & \mathbf{q} \end{array}\right] \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \right)
$$

$$
= \left(\mathcal{F}_2 \left[\begin{array}{c|c} \mathbf{p} & \\ \hline & \mathbf{q} \end{array}\right] T_2^0 \right) \boxplus \left(\mathcal{F}_2 \left[\begin{array}{c|c} \mathbf{p} & \\ \hline & \mathbf{q} \end{array}\right] T_2^1 \right),
$$

where $\{T^{\alpha_1}\}_{\alpha_1=0}^{1}$ are dyadic shifts. Using this construction for all 2^{k-2} complementary pairs ($\alpha_{[k-2]} = 0, 1, \ldots, 2^{k-2} - 1$), we obtain

$$
{}^{2^{n+1}}\mathcal{FP}_{(\alpha_{[n]},\alpha_{n+1})} = \boxplus_{\alpha_{[n]}=0}^{2^n-1} \left(\mathcal{F}_2 \left[\begin{array}{c|c} \mathbf{p} & \\ \hline & \mathbf{q} \end{array} \right] T_2^{\alpha_n} \right)
$$

$$
= \boxplus_{\alpha_{[n]}=0}^{2^n-1} \left(\mathcal{F}_2 \left[\begin{array}{c|c} \mathrm{Pr}_{(\alpha_{[n-1]},0)}(t) & \\ \hline & \mathrm{Pr}_{(\alpha_{[n-1]},1)}(t) \end{array} \right] T_2^{\alpha_n} \right). \quad (2)
$$

Repetition of this construction yields the Fourier-Prometheus matrix ${}^{2^{n+1}}\mathcal{FP}$ of size $2^{n+1} \times 2^{n+1}$.

Our new PONS construction uses in (2) three parametric unitary matrices

$$
\mathcal{U}_2(\beta, \varphi, \gamma) = \left[\begin{array}{cc} e^{i(\beta+\gamma)} \cos\varphi & e^{i(\beta-\gamma)} \sin\varphi \\ e^{-i(\beta-\gamma)} \sin\varphi & -e^{-i(\beta+\gamma)} \cos\varphi \end{array} \right]
$$

instead of $\mathcal{F}_2$:

$$
{}^{2^{n+1}}\mathcal{FP}_{(\alpha_{[n]},\alpha_{n+1})}(\vec{\beta}_{n+1}, \vec{\varphi}_{n+1}, \vec{\gamma}_{n+1}) = \boxplus_{\alpha_{[n]}=0}^{2^n-1} \Big(\mathcal{U}(\beta_{n+1}, \varphi_{n+1}, \gamma_{n+1})
$$

$$
* \left[\begin{array}{c|c} \mathrm{Pr}_{(\alpha_{[n-1]},0)}(t|\vec{\beta}_n, \vec{\varphi}_n, \vec{\gamma}_n) & \\ \hline & \mathrm{Pr}_{(\alpha_{[n-1]},1)}(t|\vec{\beta}_n, \vec{\varphi}_n, \vec{\gamma}_n) \end{array} \right] T_2^{\alpha_k} \Big), \quad (3)
$$

where

$$
\vec{\beta}_{n+1} = (\beta_1, \ldots, \beta_{n+1}),\ \vec{\varphi}_{n+1} = (\varphi_1, \ldots, \varphi_{n+1}),\ \vec{\gamma}_{n+1} = (\gamma_1, \ldots, \gamma_{n+1})
$$

are three $(n+1)$D vectors of parameters. Extra parameters $\beta_k, \varphi_k, \gamma_k$ ($k = 1, 2, \ldots, n+1$) are changed from stage to stage in this construction. The resulting matrix still has orthogonal rows and every pair is 2-complementary in the Golay-Shapiro sense.

3. PONS associated with Abelian groups

3.1 Abelian groups $\mathbf{Z}_N^n$

A natural generalization of a 2-complementary Golay pair is an N-complementary Golay N-member orthogonal set of Clifford-valued sequences $\mathbf{p}_0(t)$, ..., $\mathbf{p}_{N-1}(t)$, where $t = 0, 1, ..., N^n - 1$.

Definition 2 *Let $\mathbf{p}_0(t), \mathbf{p}_1(t), \ldots, \mathbf{p}_{N-1}(t)$ be an N-member orthogonal set of Clifford-valued sequences, where $\mathbf{p}_i(t) \in \{\varepsilon_N^k\}_{k=0}^{N-1}$, $\varepsilon_N :=$*

$e^{2\pi\mathbf{u}/N} \in \mathcal{C}la$, *$\mathcal{C}la$ is a Clifford algebra, and* $\mathbf{u}$ *is an appropriate bivector with the property* $\mathbf{u}^2 = -1$. *The sequences* $\{\mathbf{p}_i(t)\}_{i=0}^{N-1}$ *are called* N*-complementary* $\{\varepsilon_N^k\}_{k=0}^{N-1}$*-valued sequences of length* N^n *if*

$$\mathrm{COR}[\mathbf{p}_0, \mathbf{p}_0](\tau) + \ldots + \mathrm{COR}[\mathbf{p}_{N-1}, \mathbf{p}_{N-1}](\tau) = N^n \delta(\tau),$$

or $|\mathbf{p}_0(z)|^2 + |\mathbf{p}_1(z)|^2 + \ldots + |\mathbf{p}_{N-1}(z)|^2 = N^n, \quad \forall z \in \mathbb{T}_1,$ *where* $\mathbf{p}_i(z)$ *are* $\mathcal{Z}$*-transforms of* $\mathbf{p}_i(t),\ i = 0, 1, \ldots, N-1,$ *respectively.*

Let, for example, $N = 3$. Then for the group $\mathbf{Z}_3$ we define the Fourier-Clifford-Prometheus transform as the Fourier-Clifford transform

$$^{3^1}\mathcal{FCP} := {}^{3^1}\mathcal{FC} = \begin{bmatrix} \mathrm{Pr}_0(t) \\ \mathrm{Pr}_1(t) \\ \mathrm{Pr}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \varepsilon_3 & \varepsilon_3^2 \\ 1 & \varepsilon_3^2 & \varepsilon_3 \end{bmatrix}.$$

For the group $\mathbf{Z}_3^2$ we define the Fourier-Clifford-Prometheus transform using the classical PONS-construction (2) by

$$^{3^2}\mathcal{FCP}_{\alpha_1,\alpha_2} = \boxplus_{\alpha_1=0}^{2} \left(\mathcal{FC}_3 \left[\begin{array}{c|c|c} \mathrm{Pr}_0(t) & & \\ \hline & \mathrm{Pr}_1(t) & \\ \hline & & \mathrm{Pr}_2(t) \end{array} \right] T_3^{\alpha_1} \right)$$

$$= \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & & & & & & \\ 1 & \varepsilon_3 & \varepsilon_3^2 & & & & & & \\ 1 & \varepsilon_3^2 & \varepsilon_3 & & & & & & \\ \hline & & & 1 & 1 & 1 & & & \\ & & & 1 & \varepsilon_3 & \varepsilon_3^2 & & & \\ & & & 1 & \varepsilon_3^2 & \varepsilon_3 & & & \\ \hline & & & & & & 1 & 1 & 1 \\ & & & & & & 1 & \varepsilon_3 & \varepsilon_3^2 \\ & & & & & & 1 & \varepsilon_3^2 & \varepsilon_3 \end{array} \right] \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & & & & & & \\ & & & 1 & \varepsilon_3 & \varepsilon_3^2 & & & \\ & & & & & & 1 & \varepsilon_3^2 & \varepsilon_3 \\ \hline & & & 1 & 1 & 1 & & & \\ & & & & & & 1 & \varepsilon_3 & \varepsilon_3^2 \\ 1 & \varepsilon_3^2 & \varepsilon_3 & & & & & & \\ \hline & & & & & & 1 & 1 & 1 \\ 1 & \varepsilon_3 & \varepsilon_3^2 & & & & & & \\ & & & 1 & \varepsilon_3^2 & \varepsilon_3 & & & \end{array} \right]$$

$$= \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & \varepsilon_3 & \varepsilon_3^2 & 1 & \varepsilon_3^2 & \varepsilon_3 \\ 1 & 1 & 1 & \varepsilon_3 & \varepsilon_3^2 & 1 & \varepsilon_3^2 & \varepsilon_3 & 1 \\ 1 & 1 & 1 & \varepsilon_3^2 & 1 & \varepsilon_3 & \varepsilon_3 & 1 & \varepsilon_3^2 \\ \hline 1 & \varepsilon_3^2 & \varepsilon_3 & 1 & 1 & 1 & 1 & \varepsilon_3 & \varepsilon_3^2 \\ \varepsilon_3^2 & \varepsilon_3 & 1 & 1 & 1 & 1 & \varepsilon_3 & \varepsilon_3^2 & 1 \\ \varepsilon_3 & 1 & \varepsilon_3^2 & 1 & 1 & 1 & \varepsilon_3^2 & 1 & \varepsilon_3 \\ \hline 1 & \varepsilon_3 & \varepsilon_3^2 & 1 & \varepsilon_3^2 & \varepsilon_3 & 1 & 1 & 1 \\ \varepsilon_3 & \varepsilon_3^2 & 1 & \varepsilon_3^2 & \varepsilon_3 & 1 & 1 & 1 & 1 \\ \varepsilon_3^2 & 1 & \varepsilon_3 & \varepsilon_3 & 1 & \varepsilon_3^2 & 1 & 1 & 1 \end{array} \right] = \left[\begin{array}{c} \mathrm{Pr}_{(0,0)}(t) \\ \mathrm{Pr}_{(0,1)}(t) \\ \mathrm{Pr}_{(0,2)}(t) \\ \hline \mathrm{Pr}_{(1,0)}(t) \\ \mathrm{Pr}_{(1,1)}(t) \\ \mathrm{Pr}_{(1,2)}(t) \\ \hline \mathrm{Pr}_{(2,0)}(t) \\ \mathrm{Pr}_{(2,1)}(t) \\ \mathrm{Pr}_{(2,2)}(t) \end{array} \right],$$

where $\{T^{\alpha_1}\}_{\alpha_1=0}^{2}$ are 3-cyclic shift operators. After $n+1$ iterations we obtain the following Fourier-Clifford-Prometheus transform on the group

$\mathbf{Z}_3^{n+1}$:

$$^{3^{n+1}}\mathcal{FCP}_{(\alpha_{[n]},\alpha_{n+1})} = \boxplus_{\alpha_{[n]}=0}^{3^n-1} \Big(\mathcal{FC}_3 * \begin{bmatrix} \mathrm{Pr}_{(\alpha_{[n-1]},0)}(t) & & \\ & \mathrm{Pr}_{(\alpha_{[n-1]},1)}(t) & \\ & & \mathrm{Pr}_{(\alpha_{[n-1]},2)}(t) \end{bmatrix} T_3^{\alpha_n}\Big).$$

The same expression is true for the Fourier-Clifford-Prometheus transform on the group $\mathbf{Z}_N^n$:

$$^{N^{n+1}}\mathcal{FCP}_{(\alpha_{[n]},\alpha_{n+1})} = \boxplus_{\alpha_{[n]}=0}^{N^{[n]}-1} \Big(\mathcal{FC}_N * \begin{bmatrix} \mathrm{Pr}_{(\alpha_{[n-1]},0)}(t) & & & \\ & \mathrm{Pr}_{(\alpha_{[n-1]},1)}(t) & & \\ & & \ddots & \\ & & & \mathrm{Pr}_{(\alpha_{[n-1]},N-1)}(t) \end{bmatrix} T_N^{\alpha_n}\Big),$$

where $\mathcal{FC}_N$ is the Fourier-Clifford transform on the group $\mathbf{Z}_N$,

$$\{T^{\alpha_1}\}_{\alpha_1=0}^{N-1}$$

are N-cyclic shift operators.

3.2 Abelian groups $\mathbf{Z}_{N_1} \oplus \mathbf{Z}_{N_2} \oplus \ldots \oplus \mathbf{Z}_{N_n}$

Let $\mathbf{Z}_{N_1} \oplus \mathbf{Z}_{N_2} \oplus \ldots \oplus \mathbf{Z}_{N_n}$ be an Abelian group, where $N_1, N_2, \ldots, N_n$ are positive integers. The classical Fourier-Prometheus transforms are generated by the Fourier-Walsh transform $\mathcal{F}_2$ and by dyadic shifts. Fourier-Clifford-Prometheus transforms associated with $\mathbf{Z}_N^n$ are generated by the Fourier-Clifford transform $\mathcal{FC}_N$ of the group $\mathbf{Z}_N$ and by N-ary shifts. We shall generate new Fourier-Clifford-Prometheus transforms associated with Abelian groups $\mathbf{Z}_{N_1} \oplus \mathbf{Z}_{N_2} \oplus \ldots \oplus \mathbf{Z}_{N_n} \oplus \mathbf{Z}_{N_{n+1}}$ by using the set of Fourier-Clifford transforms $\mathcal{FC}_{N_1}, \mathcal{FC}_{N_2}, \ldots, \mathcal{FC}_{N_n}, \mathcal{FC}_{N_{n+1}}$. For example, the group $\mathbf{Z}_2 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_4$ requires three Fourier-Clifford transforms

$$\mathcal{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathcal{F}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \varepsilon_3 & \varepsilon_3^2 \\ 1 & \varepsilon_3^2 & \varepsilon_3 \end{bmatrix}, \quad \mathcal{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \varepsilon_4^1 & \varepsilon_4^2 & \varepsilon_4^3 \\ 1 & \varepsilon_4^2 & 1 & \varepsilon_4^2 \\ 1 & \varepsilon_4^3 & \varepsilon_4^2 & \varepsilon_4^1 \end{bmatrix}.$$

Let us consider the group $\mathbf{Z}_2 \oplus \mathbf{Z}_3$. $\mathcal{FCP}_2 = \mathcal{FC}_2 = \begin{bmatrix} \mathrm{Pr}_0(t) \\ \mathrm{Pr}_1(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. We define the Fourier-Clifford-Prometheus transform associated with the Abelian group $\mathbf{Z}_2 \oplus \mathbf{Z}_3$ by using the classical PONS construction

$$
\begin{gathered}
{}^{2\cdot 3}\mathcal{FCP}_{(\alpha_1,\alpha_2)} \\
= \boxplus_{\alpha_1=0}^{1} \left(\mathcal{FC}_3 \left[\begin{array}{c|c|c} \mathrm{Pr}_{\langle(\alpha_1,0)\rangle_2}(t) & & \\ \hline & \mathrm{Pr}_{\langle(\alpha_1,1)\rangle_2}(t) & \\ \hline & & \mathrm{Pr}_{\langle(\alpha_1,2)\rangle_2}(t) \end{array} \right] T_3^{\alpha_1} \right),
\end{gathered}
$$

where $\langle(\alpha_1,\beta_2)\rangle_2 := (\alpha_1,\beta_2) \bmod 2$. Therefore,

$$
\mathcal{FCP}_{2\cdot 3} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & & & \\ 1 & \varepsilon_3 & \varepsilon_3^2 & & & \\ 1 & \varepsilon_3^2 & \varepsilon_3 & & & \\ \hline & & & 1 & 1 & 1 \\ & & & 1 & \varepsilon_3 & \varepsilon_3^2 \\ & & & 1 & \varepsilon_3^2 & \varepsilon_3 \end{array} \right] \left[\begin{array}{cc|cc|cc} 1 & 1 & & & & \\ & & 1 & -1 & & \\ & & & & 1 & 1 \\ \hline & & 1 & 1 & & \\ & & & & 1 & -1 \\ 1 & -1 & & & & \end{array} \right]
$$

$$
= \left[\begin{array}{cc|cc|cc} 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & \varepsilon_3 & \varepsilon_3 & \varepsilon_3^2 & \varepsilon_3^2 \\ 1 & 1 & \varepsilon_3 & -\varepsilon_3 & \varepsilon_3^2 & \varepsilon_3^2 \\ \hline 1 & -1 & 1 & 1 & 1 & -1 \\ \varepsilon_3^2 & -\varepsilon_3^2 & 1 & 1 & \varepsilon_3 & -\varepsilon_3 \\ \varepsilon_3 & -\varepsilon_3 & 1 & 1 & \varepsilon_3^2 & -\varepsilon_3^2 \end{array} \right] = \left[\begin{array}{c} \mathrm{Pr}_{(0,0)}(t) \\ \mathrm{Pr}_{(0,1)}(t) \\ \mathrm{Pr}_{(0,2)}(t) \\ \hline \mathrm{Pr}_{(1,0)}(t) \\ \mathrm{Pr}_{(1,1)}(t) \\ \mathrm{Pr}_{(1,2)}(t) \end{array} \right]. \qquad (4)
$$

We design Fourier-Clifford-Prometheus transforms associated with the Abelian groups $\mathbf{Z}_{N_1} \oplus \mathbf{Z}_{N_2} \oplus \ldots \oplus \mathbf{Z}_{N_{n+1}}$ by the same classical PONS construction

$$
\begin{gathered}
{}^{N^{[n+1]}}\mathcal{FCP}_{(\alpha_{[n]}\alpha_{n+1})} \\
= \boxplus_{\alpha_{[n]}=0}^{N^{[n]}-1} \left(\mathcal{FC}_{N_{n+1}} \left[\begin{array}{c|c|c} \mathrm{Pr}_{(\alpha_{[n-1]},\langle 0\rangle_{N_n})} & & \\ \hline & \ddots & \\ \hline & & \mathrm{Pr}_{(\alpha^{(n-1)},\langle N_{n+1}-1\rangle_{N_n})} \end{array} \right] T_{N_{n+1}}^{\alpha_n} \right)
\end{gathered}
$$

where $\mathcal{F}_{N_{n+1}}$ is the Fourier-Clifford transform on the group $\mathbf{Z}_{N_{n+1}}$, $\langle \alpha_n + \beta_{n+1}\rangle_{n+1} := (\alpha_n + \beta_{n+1}) \bmod N_{n+1}$, $\alpha_{[n]} := (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $N^{[n]} := N_1 N_2 \cdots N_n$, $(\alpha_{[n]}, \beta_{n+1}) := (\alpha_1, \ldots, \alpha_n, \beta_{n+1})$, and, hence,

$$
\langle(\alpha_{[n]}, \beta_{n+1})\rangle_n := (\alpha_1, \ldots, \alpha_n, \beta_{n+1}) \bmod N_n.
$$

4. Fast Fourier-Prometheus Transforms

4.1 Radix-2 Fast Transforms

Let us return to the Fourier-Clifford-Prometheus transform

$$\mathcal{FP}_{2^2} = \left[\begin{array}{c} \mathrm{Pr}_{(0,0)}(t) \\ \mathrm{Pr}_{(0,1)}(t) \\ \hline \mathrm{Pr}_{(1,0)}(t) \\ \mathrm{Pr}_{(1,1)}(t) \end{array}\right] = \left[\begin{array}{rr|rr} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & 1 & \\ & & & -1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right] \left[\begin{array}{rr|rr} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & 1 & \\ & & & -1 \end{array}\right]$$

$$= \Delta_0 \Pi_4 (\mathcal{F}_2 \otimes \mathcal{F}_2) \Delta_0, \qquad (5)$$

where $\Delta_0 := \mathbf{diag}(\mathrm{Pr}_0(t)) = \mathbf{diag}(\mathrm{Pr}_\alpha(0))$ is a diagonal matrix and Π_4 is a special permutation matrix. From this expression we see that Prometheus functions up to constant factor are modulated Walsh functions:

$$\mathrm{Pr'}_{(\alpha_1,\alpha_2)}(t_1,t_2) = (-1)^{\alpha_1\alpha_2}\Big[\mathrm{Wal}_{(\alpha_1,\alpha_2)}(t_1,t_2)(-1)^{t_1t_2}\Big],$$

where $(-1)^{\alpha_1\alpha_2}$ and $(-1)^{t_1t_2}$ are the so-called Shapiro multipliers, and $\mathrm{Wal}_{(\alpha_1,\alpha_1)}(t_1,t_2) = (-1)^{\alpha_1 t_1 \oplus_2 \alpha_2 t_2}$. The same result is true in the general case for the Fourier-Clifford-Prometheus $(2^n \times 2^n)$-transform $\mathcal{FP}_{2^n} = \Delta_0 \Pi_{2^n}(\mathcal{F}_2 \otimes \mathcal{F}_2 \otimes \cdots \otimes \mathcal{F}_2)\Delta_0$, where Π_{2^n} is a special permutation matrix and $\Delta_0 = \mathbf{diag}(\mathrm{Pr}_0(t)) = \mathbf{diag}(\mathrm{Pr}_\alpha(0))$ is the diagonal matrix whose diagonal elements form the Shapiro (± 1)-multipliers. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the binary representation of the number in the αth row of Δ_0, where $\alpha_i \in \mathbf{Z}_2$, then for diagonal elements $\Delta_{\alpha,\alpha}$ we have the expression $\Delta_{\alpha,\alpha} = (-1)^{\sum_{i=1}^{n-1}\alpha_i\alpha_{i+1}}$. The quantity $b(\alpha) = \sum_{i=1}^{n-1}\alpha_i\alpha_{i+1}$ is the number of occurrences of the block $B = (11)$ in the binary representation of α, $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. For this reason the Fourier-Clifford-Prometheus transform has the Cooley-Tukey fast algorithm

$$\mathcal{FP}_{2^n} = \Delta_0 \Pi_{2^n}\Big[CT^1_{2^n} CT^2_{2^n} \cdots CT^n_{2^n}\Big]\Delta_0, \qquad (6)$$

where $CT^i_{2^n} := I_2 \otimes \ldots \otimes \mathcal{F}_2 \otimes \ldots \otimes I_2$ for $i = 1, 2, \ldots, n$ are the so-called *Cooley-Tukey sparse matrices.*

Now we can prove that an analogous result is true for Davis-Jedwab Clifford-valued sequences. Let $\mathbf{MC}_{2^h} = \{\varepsilon^k_{2^h}\}_{k=0}^{2^h-1}$ be the multiplicative cyclic group of 2^hth roots of unity and ε_{2^h} be a 2^hth primitive root in a

Clifford algebra $\mathcal{C}la$. Let $(c_1, c_2, \ldots, c_n) \in \mathbf{Z}_{2^h}^n = \mathbf{Z}_{2^h} \oplus \mathbf{Z}_{2^h} \oplus \ldots \oplus \mathbf{Z}_{2^h}$, be an nD vector of parameters over $\mathbf{Z}_{2^h}$, where $\mathbf{Z}_{2^h}^n$ is a set of nD vectors (labels). Let

$$\mathcal{FC}_2(\varepsilon_{2^h}^{c_k}) := \begin{bmatrix} 1 & \varepsilon_{2^h}^{c_k} \\ 1 & -\varepsilon_{2^h}^{c_k} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \varepsilon_{2^h}^{c_k} \end{bmatrix}, \quad k = 1, 2, \ldots, n, \tag{7}$$

be a set of (2×2)-matrices. Then the tensor product of these matrices

$$\mathcal{FCPDJ}_{2^n}^{(c_1, c_2, \ldots, c_n)} := \Delta_0 \Pi_{2^n} \Big(\mathcal{F}_2(\varepsilon_{2^h}^{c_1}) \otimes \mathcal{F}_2(\varepsilon_{2^h}^{c_2}) \otimes \cdots \otimes \mathcal{F}_2(\varepsilon_{2^h}^{c_n}) \Big) \Delta_0 \tag{8}$$

gives us new multi-parametric Fourier-Prometheus transforms with fast Cooley-Tukey algorithm:

$$\mathcal{FCPDJ}_{2^n}^{(c_1, c_2, \ldots, c_n)} = \Delta_0 \Pi_{2^n} \Big[CT_{2^n}^1(\varepsilon_{2^h}^{c_1}) CT_{2^n}^2(\varepsilon_{2^h}^{c_2}) \cdots CT_{2^n}^n(\varepsilon_{2^h}^{c_n}) \Big] \Delta_0, \tag{9}$$

where $CT_{2^n}^k(\varepsilon_{2^h}^{c_k}) := \Big[I_2 \otimes \ldots \otimes \mathcal{F}_2(\varepsilon_{2^h}^{c_k}) \otimes \ldots \otimes I_2 \Big]$, and $k = 1, 2, \ldots, n$.

4.2 Radix-N Fourier-Prometheus transforms

Let us consider the case of $\mathbf{Z}_3^2$. In this case $\mathcal{FCP}_{3^2} = \Delta_0 \Pi_9 (\mathcal{F}_3 \otimes \mathcal{F}_3) \Delta_0$, where $\Delta_0 = \mathbf{diag}\{\mathrm{Pr}_{(0,0)}(t_1, t_2)\} = \mathbf{diag}\{\mathrm{Pr}_{(t_1,t_2)}(0,0)\}$ and Π_9 is a special permutation matrix. From this expression we see that Prometheus functions up to constant factor are modulated Chrestenson-Clifford sequences (i.e., Clifford-valued characters of the group $\mathbf{Z}_3^2$):

$$\mathrm{Pr'}_{(\alpha_1,\alpha_2)}(t_1, t_2) = \mathrm{Pr}_{(\alpha_1,\alpha_2)}(0,0) \Big[\mathrm{Ch}_{(\alpha_1,\alpha_2)}(t_1, t_2) \cdot \mathrm{Pr}_{(0,0)}(t_1, t_2) \Big]$$

$$= \varepsilon_3^{\alpha_1 \alpha_2} \Big[\mathrm{Ch}_{(\alpha_1,\alpha_2)}(t_1, t_2) \varepsilon_3^{t_1 t_2} \Big] = \varepsilon_3^{\alpha_1 \alpha_2} \Big[\varepsilon_3^{\alpha_1 t_1 \oplus_2 \alpha_2 t_2} \cdot \varepsilon_3^{t_1 t_2} \Big],$$

where

$$\mathrm{Pr}_{(\alpha_1,\alpha_2)}(0,0) = \varepsilon_3^{\alpha_1 \alpha_2}$$

$$\mathrm{Pr}_{(0,0)}(t_1, t_2) = \varepsilon_3^{t_1 t_2},$$

and

$$\mathrm{Ch}_{(\alpha_1,\alpha_1)}(t_1, t_2) = \varepsilon_3^{\alpha_1 t_1 \oplus_2 \alpha_2 t_2}.$$

For this reason, this Fourier-Clifford Prometheus transform has the Cooley-Tukey fast algorithm $\mathcal{FP}_{3^2} = \Delta_0 \Pi_9 \Big[CT_9^1 \cdot CT_9^2 \Big] \Delta_0$, where $CT_9^1 := \mathcal{F}_3 \otimes I_3$, $CT_9^2 = I_3 \otimes \mathcal{F}_3$. The same result is true in the general case for Fourier-Clifford-Prometheus $(3^n \times 3^n)$-transforms

$$\mathcal{FP}_{3^n} = \Delta_0 \Pi_{3^n} \Big(\mathcal{F}_3 \otimes \mathcal{F}_3 \otimes \cdots \otimes \mathcal{F}_3 \Big) \Delta_0 = \Delta_0 \Pi_{3^n} \Big[CT_{3^n}^1 CT_{3^n}^2 \cdots CT_{3^n}^n \Big] \Delta_0,$$

where $CT^i_{3^n} := I_3 \otimes \ldots \otimes \mathcal{F}_3 \otimes \ldots \otimes I_3$ for $i = 1, 2, \ldots, n$ are the so-called *Cooley-Tukey sparse matrices*. Now we are ready to write the analogous expression for Fourier-Clifford-Prometheus $(N^n \times N^n)$-transforms

$$\begin{aligned} \mathcal{FP}_{N^n} &= \Delta_0 \Pi_{N^n} \Big(\mathcal{F}_N \otimes \mathcal{F}_N \otimes \cdots \otimes \mathcal{F}_N \Big) \Delta_0 \\ &= \Delta_0 \Pi_{N^n} \Big[CT^1_{N^n} CT^2_{N^n} \cdots CT^n_{N^n} \Big] \Delta_0, \end{aligned} \tag{10}$$

where $CT^i_{N^n} := I_N \otimes \ldots \otimes \mathcal{F}_N \otimes \ldots \otimes I_N$ for $i = 1, 2, \ldots, n$ are the so-called *Cooley-Tukey sparse matrices*. The same result is true in the general case for the Fourier-Clifford-Prometheus $(N^{[n]} \times N^{[n]})$-transform

$$\begin{aligned} \mathcal{FP}_{N^{[n]}} &= \Delta_0 \Pi_{N^n} \Big(\mathcal{F}_{N_1} \otimes \mathcal{F}_{N_2} \otimes \cdots \otimes \mathcal{F}_{N_n} \Big) \Delta_0 \\ &= \Delta_0 \Pi_{N^n} \Big[CT^1_{N^{[n]}} CT^2_{N^{[n]}} \cdots CT^n_{N^{[n]}} \Big] \Delta_0, \end{aligned} \tag{11}$$

where $\Delta_0 := \mathbf{diag}(\mathrm{Pr}_0(t)) = \mathbf{diag}(\mathrm{Pr}_\alpha(0))$ is a diagonal matrix and Π_{N^n} a permutation matrix, and $CT^i_{N^{[n]}} := I_{N_1} \otimes \ldots \otimes \mathcal{F}_{N_i} \otimes \ldots \otimes I_{N_n}$ for $i = 1, 2, \ldots, n$ are the so-called *Cooley-Tukey sparse matrices*.

5. Conclusions

We have shown how Clifford algebras can be used to formulate a new unified approach to so-called generalized *Fourier-Clifford-Prometheus transforms*. It is based on a new generalized FCPT-generating construction. This construction has a rich algebraic structure that supports a wide range of fast algorithms. This construction is associated not with the triple $\Big(\mathbf{Z}_2^n, \mathcal{F}_2, \mathbf{C}\Big)$, but rather with other groups instead of $\mathbf{Z}_2^n$, other unitary transforms instead of $\mathcal{F}_2$, and other algebras (Clifford algebras) instead of the complex field $\mathbf{C}$.

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