Complementary Pairs of Multilinear Polynomials

Harold Shapiro
Prometheus Inc., Sweden

Abstract

Denote by T the unit circumference in the complex plane C, and by T^d the corresponding d-dimensional torus in C^d. By M_d we denote the set of polynomials in d variables $z_1, \dots, z_d$ with complex coefficients which have degree at most one with respect to each variable. Thus M_d is a vector space of dimension $D = 2^d$ over C. By N_d we denote the subset of M_d for which all coefficients are from the set $-1, 1$. Two elements f and g of N_d are said to constitute a complementary pair if $|f|^2 + |g|^2 = 2D$ at all points of T^d. It is not hard to show, on the basis of a recursive construction introduced by the speaker in 1951, that such pairs exist for every non-negative integer d. From a complementary pair f, g others may be derived by some simple transformations such as permuting variables, or replacing some of the variables z_n with $-z_n$. The main aim of the research reported here is to test the conjecture that all complementary pairs are derivable from the "standard" one (given by the above-mentioned recursive construction) using such transformations (which form a group whose precise definition will of course be presented). Thus far this conjecture has been established only for d not exceeding 4. The main interest of complementary pairs is that, if one substitutes t, t^2 , t^4, …, $t^{D/2}$ for z_1 , z_2 , z_3, ..., z_d respectively one obtains a pair of polynomials in t of degree $D - 1$ with coefficients from $-1, 1$ whose squared moduli on T sum to $2D$ at every point. Such pairs (sometimes called Golay complementary pairs) are important for various applications; multilinear complementary pairs are thus a kind of "high ground" for construction of Golay pairs. Our main results so far are these: 1) If f, g are a complementary pair they are irreducible as elements of $C(z_1, \dots, z_d)$. This implies that if f in N_d admits a compementary "mate" g, then g is unique modulo some trivial normalizations. 2) If f ,g are a complementary pair from N_d and d is even then $|f|^2$ and $|g|^2$ each take the constant value D at all D points of T^d where each variable z_n equals -1 or 1. This can be stated in other equivalent ways, for example that the coefficient sequence of f (and of g) with a certain natural ordering has a Hadamard transform of constant modulus and thus comes from a "bent" Boolean function (these are much studied in coding theory). However, not all bent Boolean functions lead to complementary pairs. 3) There are results concerning existence and nonexistence of "complementable" elements f of N_d invariant with respect to various groups of permutations of the variables.