Some Plots of Bessel Functions of Two Variables

F.A. Grünbaum

Department of Mathematics
University of California
Berkeley, CA 94720

ABSTRACT. For a finite reflection group G there is a rich theory developed by Dunkl, Heckman and Opdam leading to the notion of a commuting set of Bessel differential operators. These systems play an important role in the study of Calogero–Moser systems and other problems of physical interest. When G acts on the real line one recovers the usual Bessel function with a well known power series expansion at the origin. We obtain some such expansions in the case of $G = A_2$ acting in the plane and we use these to produce plots of some of these functions.

1. Introduction

Special functions such as Gauss' hypergeometric series and the Bessel functions have played a crucial role in math-physics for a very long time, and many branches of engineering and technology are good examples of such use. The appearance of symbolic/numerical/graphics packages such as Macsyma, Maple, Mathematica, and others has put plots of these functions at the fingertips of many users in applied fields.

The pathbreaking work of Dunkl, Heckman and Opdam, de Jeu and others has produced a rich theory of Bessel functions of several variables which extends in a natural way the one dimensional situation. There are also several versions of Gauss' function in the case of several variables, and a large program dealing with polynomials has been developed by Macdonald, Koornwinder and Cherednik, as well as others.

The author was supported in part by NSF Grant # DMS94-00097 and by AFOSR under Contract FDF49620-96-1-0127.

These functions have arisen at times in connection with integrable systems or as spherical functions for appropriate symmetric spaces. One can wonder if these relatively recent mathematical objects will eventually have an impact on technology comparable to the one variable case, and how much they will penetrate the engineering literature.

There are at least three interrelated answers to the question above:

a) this will happen only if interesting "down-to-earth" problems ever get solved in terms of these new functions.
b) the amount of heavy going mathematical sophistication needed to go through this new material makes it unlikely that enough "applied math" people will take the time to learn about these new tools and test their applicability.
c) maybe some few plots will not hurt. After all, the best way to see the usefulness of sines and cosines—as well as Bessel functions, elliptic functions, and many other such gems—is to view some few graphs of them. Keep in mind that whether we like it or not, more and more engineering students will be trained in front of a screen with increasing graphical capabilities.

The goal of this note is rather modest. I want to show some of these plots and hopefully provoke someone into doing a better job. Given the quality of the graphs that I show this should not be too much of a challenge. I have not seen any such graphs, or even some of the power series expressions that I will use to compute with, in the literature. It is clear that for computational purposes eventually one would like something smarter than just power series, like piecewise rational approximation or similar things. This can wait until we see if these new functions get used enough so as to warrant such an effort.

2. The eigenvalue problem

On the plane with coordinates (a, b) consider the operators op_1 and op_2 given by

$$
\begin{aligned}
op_1(f) &= \frac{d^2 f}{db^2} + \frac{d^2 f}{da^2} \\
&+ k\left(\frac{1}{(\sqrt{3}b+3a)^2} + \frac{1}{(\sqrt{3}b-3a)^2} + \frac{1}{12b^2}\right) f
\end{aligned}
$$

$$
\begin{aligned}
op_2(f) &= \frac{27k\left(\frac{1}{(\sqrt{3}b-3a)^2} - \frac{1}{(\sqrt{3}b+3a)^2}\right)\frac{df}{db}}{2\sqrt{3}} \\
&+ \frac{d^3 f}{da^3} - 3\frac{d^3 f}{da\,db^2} \\
&+ \frac{3k\left(\frac{1}{(\sqrt{3}b+3a)^2} + \frac{1}{(\sqrt{3}b-3a)^2} - \frac{1}{6b^2}\right)\frac{df}{da}}{2}.
\end{aligned}
$$

Here k is an arbitray parameter.

These operators are invariant under the operations 1) b goes into $-b$ as well as 2) (a, b) goes into $(-a/2 + \sqrt{3}/2b, \sqrt{3}/2a + b/2)$. The first one is a reflection across the a axis, the second one a reflection across the axis given by the vector $(-\sqrt{3}, 1)$. The same is true if one considers one more reflection, across the axis making an angle of -60 degrees with the a axis. There is then a three-fold symmetry, and since these reflections generate a group of six elements (the symmetric group on three symbols) the full group of symmetries has order six.

The operators commute and we define the Bessel function as the function made symmetric by adding six terms of the form

$$f(a, b, s_1, s_2) = e^{s_1 a + s_2 b}(1 + \text{small at infinity})$$

that solves the system

$$op_1(f) = \lambda_1 f$$

and

$$op_2(f) = \lambda_2 f.$$

This clearly requires $\lambda_1 = s_1^2 + s_2^2$ and $\lambda_2 = s_1^3 - 3s_1 s_2^2$.

Put $a = r\cos t$ and $b = r\sin t$ and put $x = \cos 3t$ and make the analoguous change of variables on the spectral side $s_1 = \beta\cos s$, $s_2 = \beta\sin s$, $y = \cos 3s$.

For later use, introduce p as any root of the equation

$$k = -\frac{4p^2 - 12p}{3}.$$

Conjugate the resulting operators by the factor

$$r^p(1 - x^2)^{p/6}$$

i.e. define opp_i by

$$r^p(1 - x^2)^{p/6}\ opp_i = op_i\ r^p(1 - x^2)^{p/6}$$

to get new operators opp_i $(i = 1, 2)$ given by

$$-\frac{9\frac{d^2f}{dx^2}(x-1)(x+1)}{r^2} - \frac{3\frac{df}{dx}(2p+3)x}{r^2} + \frac{\frac{df}{dr}(2p+1)}{r} + \frac{d^2f}{dr^2}$$

and

$$\begin{aligned}
-\frac{3\frac{df}{dx}(35x^2 - 4p^2 + 6p - 17)}{r^3} &+ \frac{9\frac{d^2f}{drdx}(6x^2 + 2p - 3)}{r^2} \\
-\frac{27\frac{d^3f}{dx^3}(x-1)^2(x+1)^2}{r^3} &+ \frac{27\frac{d^3f}{drdx^2}(x-1)x(x+1)}{r^2} \\
-\frac{135\frac{d^2f}{dx^2}(x-1)x(x+1)}{r^3} &- \frac{9\frac{d^3f}{dr^2dx}(x-1)(x+1)}{r} \\
-\frac{3\frac{d^2f}{dr^2}x}{r} &+ \frac{3\frac{df}{dr}x}{r^2} + \frac{d^3f}{dr^3}x
\end{aligned}$$

respectively.

Consider now the resulting equations

$$opp_1\ f = \beta^2 f$$

$$opp_2\ f = \beta^3 y f.$$

It is easy to see that they admit solutions of the form

$$1 + \sum c_{ijkl} r^{2i} (r^3 x)^j \beta^{2k} (\beta^3 y)^l$$

with $c_{ijkl} = c_{klij}$.

In particular we get the well known fact that the Bessel functions are symmetric in the interchange between "spatial" and "spectral" variables. This is then a "trivial or basic" instance of the bispectral property discussed in [DG] but for systems of partial differential operators. In this connection it is pleasing to see that the relations

$$ad^3(opp_i)(r^2) = 0 \qquad i = 1, 2$$

and

$$ad^4(opp_i)(r^3 x) = 0 \qquad i = 1, 2$$

hold in this case.

If we are in the simpler case when $\beta = y = 1$ we have solutions of the form

$$1 + \sum c_{ij} r^{2i+3j} x^j$$

to the equations

$$opp_1\ f = f,\ opp_2\ f = f.$$

For this case we have a simple expression for the coefficients c_{ij}, namely

$$c_{ij} = \Gamma(p)/\Gamma(p/3) 3^i \Gamma(p/3 + i + j)/(2^{2j+2i} i! j! \Gamma(p + 2i + 3j))$$

and this will be used in the plots below. These plots, made for different values of p, give an indication of f for $x = \cos 3\theta$ in $(-1, 1)$ and r in $(0, 1)$.

To get the true Bessel functions we still need to multiply these expressions by the conjugating factor

$$r^p (1 - x^2)^{p/6}$$

used above.

E. Opdam [O2] mentioned to me that M. de Jeu had made some relevant computations. I have received some notes from de Jeu [dJ2] who has obtained other expressions which could be used in producing plots of the Bessel functions.

There are also expressions for those Bessel functions in [O] that could be used for plotting purposes.

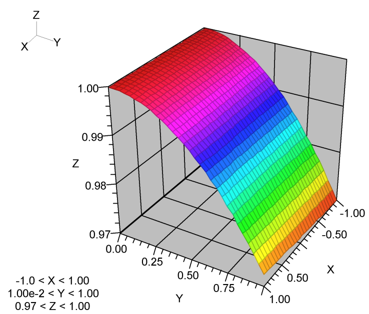

FIGURE 1. p=-19/2

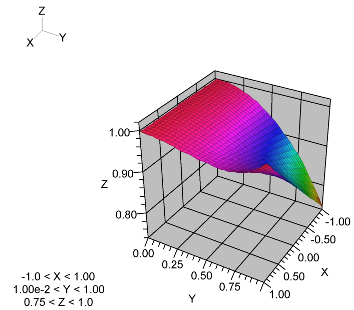

FIGURE 2. p=-5/2

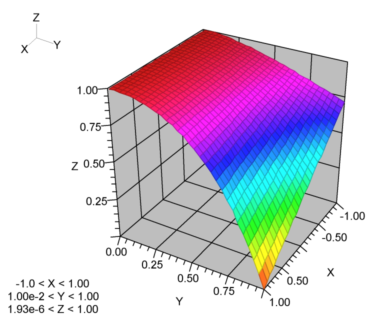

FIGURE 3. p=-3/2

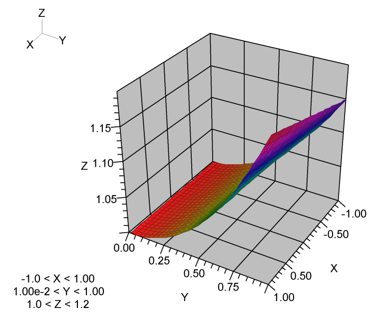

FIGURE 4. p=1/2

References

[dJ1] de Jeu, M. F. E., *The Dunkl transform*, Invent. Math. **113**, No. 1 (1993), 147–162.

[dJ2] _______________, Private communication.

[DG] Duistermaat, J., and Grünbaum, F. A., *Differential equations in the spectral parameter*, CMP **103** (1986), No. 2, 177–240.

[D] Dunkl, C., *Hankel transforms associated to finite reflection groups, hypergeometric functions in domains of positivity, Jack polynomials, and applications*, (Tampa, Florida, 1991), Contemp. Math. **138** (1992), 123–138.

[H] Heckman, G., *Integrable systems and reflection groups*, Unpublished lecture notes, Fall School on Hamiltonian Geometry, Woudschoten, 1992.

[O] Okounkov, A., and Olshanski, G., *Shifted Jack polynomials, binomial formula, and applications*, Math. Res. Letters **4** (1997), 69–78.

[O1] Opdam, E., *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group*, Compositio Math. **85** (1993), 333–373.

[O2] _______________, Private communication.